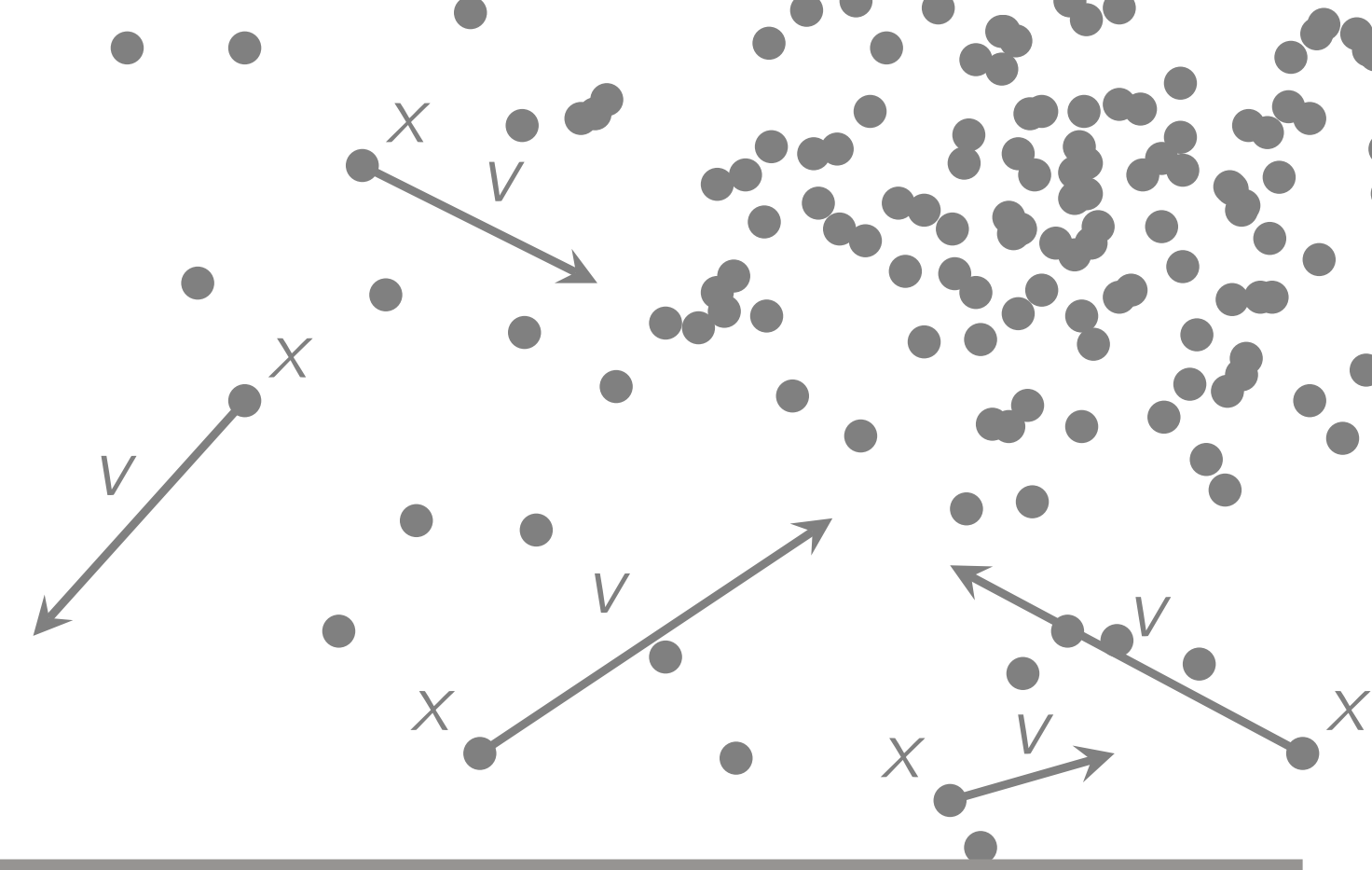


# Kinetic maximal $L^p$ -regularity

and applications to quasilinear equations

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## The problem

We study  $L^p$ -solutions  $u = u(t, x, v): [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  to linear kinetic equations of the type

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = Au + f \\ u(0) = g, \end{cases} \quad (1)$$

where  $A$  is an operator in a suitable function space  $X$  and  $f, g$  are given data. **Goal:** Characterize unique solutions  $u$  to equation (1) with  $\partial_t u + v \cdot \nabla_x u, Au \in L^p((0, T); X)$  in terms of functions spaces for the data  $f$  and  $g$ . In particular, show that the solutions to equation (1) define a semi-flow in the trace space.

If  $A$  admits such a characterization we say that  $A$  enjoys **kinetic maximal  $L^p(X)$ -regularity**.

## Divide and conquer

- The case of **vanishing initial data**, i.e.  $g = 0$ . Using singular integral theory and the solution representation given by a fundamental solution. Complicated operators can often be reduced to simpler cases.
- The **homogeneous** case, i.e.  $f = 0$ . Make sense of the temporal trace. How does the kinetic term transfer regularity from  $v$  to  $x$  on this level?

## The most important example

For  $\beta \in (0, 2]$  consider the **(fractional) Kolmogorov equation** in  $\mathbb{R}^{2n}$ ,

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\frac{\beta}{2}} u + f \\ u(0) = g. \end{cases} \quad (2)$$

The equation dictates that for strong  $L^p$ -solutions the right solution space is

$$\mathbb{E}(0, T) = \{u: u, \partial_t u + v \cdot \nabla_x u, (-\Delta_v)^{\frac{\beta}{2}} u \in L^p((0, T); L^p(\mathbb{R}^{2n}))\},$$

i.e.  $X = L^p(\mathbb{R}^{2n})$ . Using operator theoretic properties of the characteristics  $(t, x, v) \mapsto (t, x + tv, v)$  we can prove

$$\mathbb{E}(0, T) \hookrightarrow C([0, T]; L^p(\mathbb{R}^{2n})),$$

whence  $X_\gamma$ , the trace space of  $\mathbb{E}(0, T)$ , is well-defined and  $\mathbb{E}(0, T) \hookrightarrow C([0, T]; X_\gamma)$ . A theorem of Bouchut (2002) shows that

$$\mathbb{E}(0, T) = \mathbb{E}(0, T) \cap L^p((0, T); H^{\frac{\beta}{\beta+1}, p}(\mathbb{R}^{2n})).$$

Hence, the trace space should also have some regularity in  $x$ . Indeed, using methods from harmonic analysis and the fundamental solution we prove

$$X_\gamma \cong B_{pp,v}^{\beta(1-1/p)}(\mathbb{R}^{2n}) \cap B_{pp,x}^{\frac{\beta}{\beta+1}(1-1/p)}(\mathbb{R}^{2n}).$$

In particular,  $A = -(-\Delta_v)^{\beta/2}$  admits kinetic maximal  $L^p$ -regularity in  $L^p(\mathbb{R}^{2n})$ .

## Extensions

- Lebesgue spaces with temporal weights of the form  $t^{1-\mu}$  for  $\mu \in (1/p, 1]$ . This allows to consider initial values with lower regularity and to see that solutions regularize instantaneously.
- Different exponents of integrability,  $p$  in time and  $q$  in space.
- For  $q = 2$  we characterize weak solutions to the (fractional) Kolmogorov equation, cf. [1].
- Polynomial weights  $(1 + |v|)^k$  for  $k \in \mathbb{R}$  in the spatial variable.

## More examples

The linearization of many nonlinear kinetic models leads to the **(fractional) Kolmogorov equation with variable coefficients**.

The characterization of strong  $L^p$ -solutions for the Kolmogorov equation can be extended to

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = a(t, x, v): \nabla_v^2 u + b \cdot \nabla_v u + cu + f \\ u(0) = g \end{cases}$$

provided that  $a(t, x, v) \geq \lambda > 0$ ,  $b \in L^\infty$ ,  $c \in L^\infty$  and that

- $(t, x, v) \mapsto a(t, x, v)$  is bounded and uniformly continuous **or**
- $(t, x, v) \mapsto a(t, x + tv, v)$  is bounded and uniformly continuous, see [2].

If  $a$  has a special structure we are also able to treat the case of non-uniform ellipticity.

In [3] we study **non-local** operators with variable coefficients

$$[A_{t,x,v}^a u](t, x, v) = \text{p.v.} \int_{\mathbb{R}^n} (u(t, x, v+h) - u(t, x, v)) \frac{a(t, x, v, h)}{|h|^{n+\beta}} dh,$$

where  $a$  is symmetric in  $h$ , satisfies a similar continuity property in  $(t, x, v)$  and is also Hölder continuous in  $v$ .

## An application

The precise solution theory allows to study **quasilinear kinetic partial differential equations** of the type

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = A(u)u + F(u), t > 0 \\ u(0) = u_0. \end{cases}$$

We are interested in  $L^p(X)$  solutions, that is functions such that  $\partial_t u + v \cdot \nabla_x u, A(u)u \in L^p(X)$ . Under a local Lipschitz assumption on the operators  $A$  and  $F$  we are able to prove local in time existence for all  $u_0 \in X_\gamma$  such that  $A(u_0)$  admits kinetic maximal  $L^p(X)$ -regularity.

### Two Examples:

- A quasilinear diffusion problem,  $A(w)u = \nabla_v \cdot (\kappa(w)\nabla_v u)$  for suitable functions  $\kappa$ .
- A kinetic toy model of the form  $A(w)u = -(\int_{\mathbb{R}^n} w dv) (-\Delta_v)^{\frac{\beta}{2}} u$  for  $\beta \in (0, 2]$ .

The precise characterization of the trace space is very helpful to control the nonlinearities. For example embeddings such as

$$B_{pp,v}^{2-2/p}(\mathbb{R}^{2n}) \cap B_{pp,x}^{\frac{2}{3}(1-1/p)}(\mathbb{R}^{2n}) \hookrightarrow C_0(\mathbb{R}^{2n})$$

are available for  $p > 2n + 1$ .

## Ongoing research

- Establish the kinetic maximal  $L^p$ -regularity for more operators, such as for example the kinetic Fokker-Planck equation  $Au = \Delta_v u + v \cdot \nabla_v u$ .
- Weak  $L^p$ -solutions.
- Study the local existence of solutions for more complicated quasilinear equations.
- Global in time existence for quasilinear equations. Here, one needs to incorporate a priori estimates from the kinetic De Giorgi-Nash-Moser theory.

## References

- [1] L. N. and R. Z. Kinetic maximal  $L^2$ -regularity for the (fractional) Kolmogorov equation. *Journal of Evolution Equations*, **21**, 2021.
- [2] L. N. and R. Z. Kinetic maximal  $L^p$ -regularity with temporal weights and application to quasilinear kinetic diffusion equations. *Journal of Differential Equations*, **307**, 2021.
- [3] L. N. Kinetic maximal  $L^p_p(L^p)$ -regularity for the fractional Kolmogorov equation with variable density. *Nonlinear Analysis*, **214**, 2022.

