KINETIC MAXIMAL L^p-REGULARITY

with temporal weights and application to quasilinear kinetic diffusion equations

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KOLMOGOROV EQUATION

We consider the Cauchy problem

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases} \tag{1}$$

with $u = u(t,x,v)\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and data f, g. The function u models the density of moving particles at time t, position x and with velocity v given by a Wiener process.

Key points:

- Studied first by Kolmogorov in 1934 (gave a fundamental solution).
- Prototype for the Boltzmann equation.
- Degenerate - Laplacian acts in half of the variables.
- Unbounded coefficient in front of the lower order term.
- The transport part $\partial_t + v \cdot \nabla_x$ is called kinetic term.

L^p-SOLUTIONS

We are interested in L^p-solutions of the Kolmogorov equation.

- Local L^p-estimates provided in 1967 by Rothschild and Stein.
- Extended in several directions in the last 20 years.

By the methods of Folland and Stein provided in 1974 one can prove the following. For all $f \in L^p((0,T); L^p(\mathbb{R}^{2n}))$ and $g = 0$ the solution u of the Kolmogorov equation satisfies

$$\|\partial_t u + v \cdot \nabla_x u\|_p + \|\Delta_v u\|_p \leq C \|f\|_p$$

There is no control of the time-derivative, in particular, the classical theory of maximal L^p-regularity is not applicable. **Goal:** Characterize f, g in terms of function spaces such that $u \in L^p((0,T); H_v^{2,p}(\mathbb{R}^{2n}))$ and $\partial_t u + v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))$ (we write $u \in Z$).

IS THERE REGULARITY IN X?

Just before the begin of the 20th century it was proven that L^p-solutions to the kinetic equations with regularity in v admit some regularity in x, too. This development culminated in the work of Bouchut in 2002. We call this the kinetic transfer of regularity. Let $\beta \geq 0$. Bouchut proved that if $u \in L^p((0,T); H_v^{\beta,p}(\mathbb{R}^{2n}))$ is a solution of the kinetic transport equation

$$\partial_t u + v \cdot \nabla_x u = g$$

with $g \in L^p((0,T); L^p(\mathbb{R}^{2n}))$, then,

$$u \in L^p((0,T); H_x^{\frac{\beta}{\beta+1},p}(\mathbb{R}^{2n})).$$

This quantifies the regularity transfer from β-derivatives in velocity v to $\frac{\beta}{\beta+1}$-derivatives in space x. The estimate can be motivated by scaling properties of the kinetic term.

INITIAL VALUE PROBLEM

By the regularity transfer result of Bouchut we have

$$Z = Z \cap L^p((0,T); H_x^{2/3,p}(\mathbb{R}^{2n})).$$

Obviously, the trace space of Z should control regularity in x and v. However, the trace space can't be characterized by classical interpolation theory. We prove that the trace space can be characterized by an anisotropic Besov space, i.e.

$$\operatorname{tr}(Z) \cong B_{pp,v}^{2(1-1/p)}(\mathbb{R}^{2n}) \cap B_{pp,x}^{2/3(1-1/p)}(\mathbb{R}^{2n}),$$

by use of Littlewood-Paley decomposition, Fourier analysis and the fundamental solution of the Kolmogorov equation.

Recall:

$$Z = \{u\colon u, \Delta_v u, \partial_t u + v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

CONTINUITY IN TIME

Define the operator Γ on $L^p((0,T); L^p(\mathbb{R}^{2n}))$ as $\Gamma u = u(t, x + tv, v)$. Then, $\partial_t \Gamma = \Gamma(\partial_t + v \cdot \nabla_x)$. Moreover, $\Gamma(t)$ defines a strongly continuous group on $L^p(\mathbb{R}^{2n})$. Writing $u = \Gamma^{-1}(t)\Gamma(t)u$ gives $u \in C([0,T]; L^p(\mathbb{R}^{2n}))$ if $u \in Z$. An abstract argument due to Amann shows that every function $u \in Z$ is continuous with values in the trace space $\operatorname{tr}(Z)$.

KIN. MAX. L^p-REGULARITY

Main result:

For all $p \in (1, \infty)$ the Kolmogorov equation (1) admits a unique solution $u \in Z$ if and only if

- $f \in L^p((0,T); L^p(\mathbb{R}^{2n}))$
- $g \in B_{pp,v}^{2(1-1/p)}(\mathbb{R}^{2n}) \cap B_{pp,x}^{2/3(1-1/p)}(\mathbb{R}^{2n})$.

Moreover, the solution is continuous with values in the trace space $B_{pp,v}^{2(1-1/p)}(\mathbb{R}^{2n}) \cap B_{pp,x}^{2/3(1-1/p)}(\mathbb{R}^{2n})$. We say that Δ_v admits *kinetic maximal $L^p_\mu(L^p)$-regularity*.

FURTHER EXTENSIONS

We extend the results in several directions.

- We replace Δ_v by the fractional Laplacian $-(-\Delta_v)^{\frac{\beta}{2}}$ and study operators with variable coefficients $a(t,x,v)\colon \nabla_v^2 + b \cdot \nabla_v + cu$.
- We investigate different exponents of integrability p in t and q in (x,v).
- We introduce temporal weights of the form $t^{1-\mu}$ for $\mu \in (1/p, 1]$ wich allow to see instantaneous regularization nicely and reduce the initial value regularity.
- We fully characterize weak solutions in the case $p \in (1,\infty)$, $q = 2$ and $\mu \in (1/p, 1]$.

QUASILINEAR KINETIC EQ.

We use the theory of *kinetic maximal L^p-regularity* to prove short-time existence of solutions u in the class Z to the quasilinear kinetic diffusion problem

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \nabla_v \cdot (a(u)\nabla_v u) \\ u(0) = g \end{cases}$$

with g in the trace space of Z. We only need to assume $a \in C_b^2(\mathbb{R}; \mathrm{Sym}(n))$ with $a \geq \lambda \mathrm{Id}$ and $p > 4n + 2$ so that

$$\operatorname{tr}(Z) \hookrightarrow C_0(\mathbb{R}^{2n}) \cap {}_vC_0^1(\mathbb{R}^{2n}).$$

REFERENCES

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FUTURE RESEARCH

Quasilinear kinetic equations raise many interesting questions. A priori estimates and global existence are important topics to name a few. Moreover, the regularity of weak L^p-solutions needs further investigation.

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