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Kolmogorov Equations - Well-Posedness, Regularity, Asymptotics and Harnack Inequalities

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1 Introduction

In 1827 the Scottish botanist Robert Brown discovered what is nowadays known as Brownian motion. While investigating small pollen grains in a water suspension through his microscope he noticed an irregular movement of the pollen grains. In the subsequent years there were various mathematical explanations of this phenomenon proposed. For example by Albert Einstein and by Paul Langevin in the early 20th century. A modern mathematical description can be given using the notion of stochastic differential equations and the Wiener process. The Wiener process $W(t)$ is a stochastic process with independent, stationary and normally distributed increments. A particle at time t is described by its position $x(t)$ and its velocity $v(t)$. We model the irregular movement of the particle as a random fluctuation of its velocity, i.e.

$$v(t) = W(t)$$

or written in terms of stochastic calculus $\mathrm{d}v(t) = \mathrm{d}W(t)$. The position of the particle is uniquely determined by the velocity through the rule

$$\mathrm{d}x(t) = v(t)\mathrm{d}t.$$

The behavior of the particle is thus described by a system of stochastic differential equations. At least formally the law of the position of the particle is given by the integrated Wiener Process

$$x(t) = \int_0^t W(t)\mathrm{d}t + x(0).$$

Under the assumption of a spherical particle in a suspension of significantly smaller particles of spherical shape one can derive this model from physical principles up to some normalizing constant.

To every stochastic differential equation one can associate a forward Kolmogorov equation. Here this equation describes the evolution of the probability density $p(t, v, x)$ describing the state of the particle. The value of $p(t, v, x)$, at least formally, gives the probability that a particle at time t with velocity v is at the position x. The scaled forward Kolmogorov equation corresponding to the presented model of a moving particle is given by

$$\partial_t p(t, v, x) + v \cdot \nabla_x p(t, v, x) = \Delta_v p(t, v, x). \tag{1}$$

This equation is a degenerate elliptic-parabolic partial differential equation of second order. Its degeneracy is due to the fact that the Laplacian acts only in the velocity variable. It has already been studied by Andrei Kolmogorov in 1934 and therefore is also known as the Kolmogorov equation. Even though there is only diffusion in the velocity variable, solutions of the Kolmogorov equation are smooth. This is a consequence of the coupling of the position and velocity variable in the transport term $v \cdot \nabla_x$, which is an example of an interesting property of the Kolmogorov equation.

The Kolmogorov equation is the prototype of a wide class of partial differential equations of the form

$$\partial_t u(t,x) = \operatorname{div}(A(t,x)\nabla u(t,x)) + \langle b(t,x), \nabla u(t,x)\rangle \tag{2}$$

for suitable coefficients A and b, where we only assume the matrix A to be positive semidefinite. Equations of this type will be the subject of this work. Let us give an overview of the results collected in the present thesis.

In chapter 2 we are going to investigate the general class of second order degenerate elliptic-parabolic differential equations as presented in equation (2). Making use of semigroup methods, we are going to present a L^p-well-posedness result on the whole space $\mathbb{R}^N$ which goes back to the work of Baoswan Wong-Dzung in 1983. Furthermore, we are going to give a short overview of the theory of degenerate elliptic partial differential equations of second order on bounded domains. This theory has been developed by Gaetano Fichera around 1960.

In chapter 3 we are going to investigate the Cauchy problem for partial differential equations similar to the Kolmogorov equation. On the one hand, we are going to apply the results from chapter 2 to obtain an existence result. On the other hand, we are going to present two approaches which seem to be more suitable for the Kolmogorov equation. It has been shown by Andrei Kolmogorov in 1934 that the Kolmogorov equation admits a fundamental solution. We are going to derive this fundamental solution and investigate its interesting properties. Furthermore, we are going to study the Kolmogorov equation on a bounded physical domain where the velocities can attain arbitrary values in $\mathbb{R}^n$.

The results from chapter 2 and 3 show that the Kolmogorov equation can be treated by semigroup theory. Moreover, one can determine the generator of the Kolmogorov equation in $L^p(\mathbb{R}^N)$. In chapter 4 we are going to investigate the spectrum of this generator. The spectrum of the generator of a non-degenerate Kolmogorov equation has been studied first by Giorgio Metafune in 2001. These arguments can be easily adapted to the degenerate case to get some information on the spectrum of the generator of the Kolmogorov equation. In particular, we are going to show that the growth bound is equal to zero and that the corresponding semigroup is not analytic.

Using the explicit formula for the fundamental solution, we are able to study the long-time behavior of solutions to the Kolmogorov equation. The method is inspired by a similar result for the heat equation. This short exposition is the content of chapter 5.

Even though the Kolmogorov equation is not a parabolic partial differential equation, the solutions of the equation admit good regularity properties. For example, using the fundamental solution, one can see that every solution is smooth for positive times. Using the theory of hypoelliptic operators and the famous theorem of Lars Hörmander, we are going to show that every distributional solution of the Kolmogorov equation is smooth in the interior of its domain. This is due to the coupling of the position and velocity variables by the transport term $v \cdot \nabla_x$. The coupling is deeply connected to the commutator identity

$$[v\partial_x, \partial_v] = v\partial_x\partial_v - \partial_v(v\partial_x) = \partial_x.$$

We are going to make this connection more rigorous in chapter 6. Using the commutator identity, we are also going to show global bounds of solutions to the Kolmogorov equation in a suitable fractional Sobolev norm.

In the last chapter we are going to derive a Harnack inequality for the Kolmogorov equation on $\mathbb{R}^N$ by making use of the fundamental solution. The approach presented here goes back to work of Andrea Pascucci and Sergio Polidoro in 2004 and is inspired by the work of Peter Li and Shing Tung Yau in 1986. We are going to derive a differential Harnack inequality from which we deduce the Harnack inequality. Inspired by the work of Ennio De Girogi, Jürgen Moser and John Nash around 1960 it is natural to ask whether a Harnack inequality can hold under the assumption of measurable and bounded diffusion coefficients. We are going to present the positive result which was proven very recently by François Golse, Cyril Imbert, Clément Mouhot and Alexis Vasseur. Finally, we collect some notes regarding the investigation whether a weak Harnack inequality holds.

The reader who is interested in the Kolmogorov equation solely can skip chapter 2 at a first read. Section 3.1.1 is somehow fundamental to the results of section 3.1, chapter 4, chapter 5, chapter 6 and section 7.1.2. We therefore recommend to keep the results presented therein in mind throughout the whole read.

In the appendix A we present a very short introduction to semigroup theory, some helpful results on approximation and smoothing of functions and the basic definition of fractional Sobolev spaces with the help of Fourier transform. The appendix B is a collection of technical results used throughout this work. The notation used is collected in the final chapter.

2 Degenerate second order elliptic partial differential equations

2.1 A weak maximum principle for degenerate second order partial differential equations

Let $A \in C(\mathbb{R}^N, \mathbb{R}^{N\times N})$ be positive semidefinite, $b \in C(\mathbb{R}^N, \mathbb{R}^N)$ and $c \in C(\mathbb{R}^N, \mathbb{R})$ be bounded from above by $c_0 \in \mathbb{R}$. On sufficiently smooth functions we study the partial differential operator

$$Eu = \operatorname{tr}(A\nabla^2 u) + \langle b, \nabla u\rangle + cu.$$

The presented maximum principle will be used to show uniqueness of classical solutions of the Kolmogorov equation. It is taken from [Lor17].

Proposition 2.1.1. We assume that there exists a nonnegative function $\varphi \in C^2(\mathbb{R}^N)$ and constants $\lambda_0 > c_0$, $C \in \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} \varphi(x) = \infty \text{ and } E\varphi - \lambda_0\varphi \leq C.$$

Let $u \in C([0,T]\times\mathbb{R}^N)\cap C^{1,2}((0,T]\times\mathbb{R}^N)$ be a function satisfying $Eu-\partial_t u \geq 0$ in $(0,T]\times\mathbb{R}^N$ such that it holds $u(0,x_0) \leq 0$ for all $x_0 \in \mathbb{R}^N$ and that it is

$$\limsup_{|x|\to\infty}\left(\sup_{t\in[0,T]}\frac{u(t,x)}{\varphi(x)}\right) \leq 0. \tag{2.1.1}$$

Then $u \leq 0$ on $[0,T]\times\mathbb{R}^N$.

Proof. We remark that by replacing φ with $\varphi + M$ for a constant $M > \frac{C}{\lambda_0 - c_0}$ we might assume $C = 0$, i.e. $E\varphi - \lambda_0\varphi \leq 0$. Furthermore, if $c_0 > 0$, we might, by replacing $u(t,x)$ with $\exp(-c_0 t)u(t,x)$, reduce to the case $c_0 \leq 0$. This implies that we only need to consider the case $c_0 \leq 0$. We introduce the function $v(t,x) = \exp(-\lambda_0 t)u(t,x)$ and the functions

$v_k = v - k^{-1}\varphi$ for $k \in \mathbb{N}$. Clearly, it suffices to show that $v_k \leq 0$ in $[0,T] \times \mathbb{R}^N$ for all $k \in \mathbb{N}$. We calculate

$$\begin{aligned}\partial_t v_k - (E - \lambda_0) v_k &= -\lambda_0 v + \exp(-\lambda_0 t)\left(\partial_t u(t,x) - [Eu](t,x)\right) + \frac{1}{k}\left(E\varphi - \lambda_0\varphi\right) + \lambda_0 v \\ &\leq k^{-1}(E\varphi - \lambda_0\varphi) \leq 0\end{aligned}$$

in $(0,T] \times \mathbb{R}^N$. Using equation (2.1.1) and the coercivity of φ, we deduce that v_k attains a maximum at the point $(t_k, x_k) \in [0,T] \times \mathbb{R}^N$. The case $t_k = 0$ would imply that $u \leq u(t_k) = u(0) \leq 0$. It remains to consider the case $t_k \in (0,T]$. Due to the fact that v_k is two times continuously differentiable, it holds $[\partial_t v_k](t_k, x_k) \geq 0$, $[\nabla v_k](t_k, x_k) = 0$ and $[\nabla^2 v_k](t_k, x_k) \leq 0$. By lemma B.0.3, we deduce

$$[Ev_k - cv_k](t_k, x_k) = \operatorname{tr}(A\nabla^2 v_k) \leq 0$$

and consequently

$$[Ev_k - cv_k](t_k, x_k) \leq 0 \leq [\partial_t v_k](t_k, x_k) \leq [Ev_k - \lambda_0 v_k](t_k, x_k).$$

From $c(x_k) < \lambda_0$ we conclude

$$v_k(t_k, x_k) \leq 0,$$

whence $u \leq 0$ in $[0,T] \times \mathbb{R}^N$. $\square$

2.2 Well-posedness of a degenerate diffusion equation with unbounded coefficients

In this section we are going to investigate the differential operator

$$[Eu](x) = -\operatorname{div}(A(x)\nabla u) + \langle b(x), \nabla u\rangle + c(x)u(x), \tag{2.2.1}$$

$x \in \mathbb{R}^N$ for functions $u\colon \mathbb{R}^N \to \mathbb{R}$. We are going to prove that a suitable realization of this operator generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$ for $1 < p < \infty$. The results of this section are going to be used in chapter 3 to study the well-posedness of Kolmogorov equations in $\mathbb{R}^N$. We will allow two peculiarities of the differential operator. The first peculiarity is the positive semidefiniteness of the diffusion matrix and the second peculiarity is a consequence of the unbounded coefficients A, b. The results and most of the proofs are based on the article [WD83]. We are going to use several notions from semigroup theory without further explanation. Every nonstandard concept used in this section is explained in the appendix A.1.

2.2.1 Realization of the differential operator in $L^p(\mathbb{R}^N)$

Let $p \in (1,\infty)$ and $q \in (1,\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We are going to make the following assumptions on the coefficients of the differential operator E.

(A1) $A \in C^2(\mathbb{R}^N; \mathbb{R}^{N\times N})$ with bounded second derivatives, $b \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ with bounded derivatives and $c \in L^\infty(\mathbb{R}^N)$.

(A2) For all $x \in \mathbb{R}^N$ the matrix $A(x)$ is assumed to be positive semidefinite.

We set the minimal realization of E as the realization of E on $C_c^\infty(\mathbb{R}^N)$ and will denote this as $(E, C_c^\infty(\mathbb{R}^N))$ subsequently. Let us introduce the formal adjoint of E given by

$$E^T v = -\operatorname{tr}(A\nabla^2 v) - \sum_{i,j=1}^N \partial_{x_j} a_{ij} \partial_{x_i} v - \langle b, \nabla v\rangle + (c - \operatorname{div}(b))v$$

for all $v \in C_c^\infty(\mathbb{R}^N)$, where a_{ij} denote the entries of the matrix A. We note that we can also write E as

$$Eu = -\sum_{i,j=1}^N \partial_{x_i}\partial_{x_j}(a_{ij}u) + \partial_{x_i}((\partial_{x_j} a_{ij} + b_i)u) + (c - \operatorname{div}(b))u$$

so that, by partial integration, it holds $\langle Eu, v\rangle = \langle u, E^T v\rangle$ for all $u, v \in C_c^\infty(\mathbb{R}^N)$. E^T is useful if one wants to define E in the distributional sense in a shorthand way. For every

$u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ we define Eu in the distributional sense as

$$\langle Eu, \varphi \rangle = \left\langle u, E^T \varphi \right\rangle .$$

Since $E^T\varphi \in C_c(\mathbb{R}^N)$ whenever $\varphi \in C_c^\infty(\mathbb{R}^N)$, the right-hand side is well-defined for all $\varphi \in C_c^\infty(\mathbb{R}^N)$. Thus, Eu is indeed well-defined in the distributional sense for all $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. The maximal realization of E is defined on $D(\mathcal{E}_p) = \{u \in L^p(\mathbb{R}^N) \mid Eu \in L^p(\mathbb{R}^N)\}$, i.e. for every function $u \in L^p(\mathbb{R}^N)$ such that the distributional derivative Eu can be represented by a function $Eu \in L^p(\mathbb{R}^N)$. We denote this function by $\mathcal{E}_p u = Eu$. Whenever $\mathcal{E}_p u$ can be understood in the sense of classical differentiation we are going to write Eu instead of $\mathcal{E}_p u$. We introduce the, by assumption (A1) finite, constants

$$A^\infty = \max_{i,j,k,l=1,\dots,N} \|\partial_{x_i}\partial_{x_j} a_{kl}(x)\|_{\infty,\mathbb{R}^N},\ b^\infty = \max_{i,j=1,\dots,n} \|\partial_{x_i} b_j(x)\|_{\infty,\mathbb{R}^N},\ c^\infty = \|c\|_{\infty,\mathbb{R}^N}$$

for the bounds on the coefficients A, b, c. Let us sketch the course of action for the rest of this section. We want to show that the negative maximal realization is quasi-m-dispersive. From this we would be able to deduce that it is the generator of a strongly continuous, quasi-contractive positive semigroup. The quasi-dispersiveness can be obtained by careful estimating. It turns out to be more complicated to show that the maximal realization is quasi-m-dispersive. We are going to approximate E by suitable elliptic differential operator E_ε. In the elliptic case it is easier to obtain the desired result. We aim to consider the limit $\varepsilon \to 0$ in a weak sense. To do so, we need a good understanding of the operator E_ε.

We want to highlight some important properties of E. Assumption (A2) implies that, by lemma B.0.1, we are allowed to use the Cauchy-Schwarz inequality for $\langle A(x)\cdot, \cdot\rangle$ at every point $x \in \mathbb{R}^N$. Moreover, the formal adjoint E^T satisfies the assumptions (A1) and (A2) as well. This can be seen by writing the term $\mathrm{tr}(A\nabla^2 v)$ in divergence form. This will lead to an additional first order term with bounded derivatives. Useful properties of cutoff and smoothing functions are collected in the appendix A.3. These results will be used throughout this section.

2.2.2 Quasi-dispersiveness of the minimal realization

Proposition 2.2.1. For all $u \in D(\mathcal{E}_p) \cap C^2(\mathbb{R}^N)$ it holds

$$\left\langle Eu, (u^+)^{p-1} \right\rangle \geq -2M \left\langle u, (u^+)^{p-1} \right\rangle = -2M \|u^+\|^p_{p,\mathbb{R}^N}. \tag{2.2.2}$$

The constant M is chosen as $M = \max\{b^\infty, c^\infty\}$. In particular, we deduce that the negative minimal realization $(-E, C_c^\infty(\mathbb{R}^N))$ is quasi-dispersive.

Proof. To integrate by parts with negligible boundary terms, we use the cutoff functions $\eta_k(x)$ for $k \in \mathbb{N}$. We refer to section A.3 for the definition of these functions and some useful properties. To handle the integral containing $\nabla(u^+)^{p-1} = (p-1)(u^+)^{p-2}\nabla u^+$, we introduce the approximation $f_\delta = u^+((u^+)^2+\delta^2)^{\frac{p-2}{2}}$ of $(u^+)^{p-1}$ with $\delta > 0$ if $p < 2$ and $\delta = 0$ if $p \geq 2$. Here f'_δ denotes the partial derivative with respect to u^+ as an argument of the function f_δ. We write

$$\left\langle Eu, \eta_k^2 f_\delta \right\rangle = -\int_{\mathbb{R}^N} \operatorname{div}(A\nabla u)\eta_k^2 f_\delta \mathrm{d}x + \int_{\mathbb{R}^N} \langle b, \nabla u\rangle \eta_k^2 f_\delta \mathrm{d}x + \int_{\mathbb{R}^N} cu\eta_k^2 f_\delta \mathrm{d}x =: I_1 + I_2 + I_3$$

and study each term separately. The first term can be estimated as

$$\begin{aligned} I_1 &= -\int_{\mathbb{R}^N} \operatorname{div}(A\nabla u)\eta_k^2 f_\delta \mathrm{d}x = -\sum_{i=1}^N \int_{\mathbb{R}^N} \partial_{x_i}(A\nabla u)_i \eta_k^2 f_\delta \mathrm{d}x \\ &= 2\int_{\mathbb{R}^N} \left\langle A\nabla u^+, \nabla \eta_k \right\rangle \eta_k f_\delta \mathrm{d}x + \int_{\mathbb{R}^N} \left\langle A\nabla u^+, \nabla u^+ \right\rangle f'_\delta \eta_k^2 \mathrm{d}x \\ &\geq -2\int_{\mathbb{R}^N} \sqrt{\langle A\nabla\eta_k, \nabla\eta_k\rangle}\sqrt{\langle A\nabla u^+, u^+\rangle}\eta_k f_\delta \mathrm{d}x + \int_{\mathbb{R}^N} \left\langle A\nabla u^+, \nabla u^+ \right\rangle f'_\delta \eta_k^2 \mathrm{d}x \end{aligned}$$

by the Cauchy-Schwarz inequality and using $\nabla u^+ \nabla u^- = 0$. To estimate further, we calculate f'_δ as

$$f'_\delta = ((u^+)^2+\delta^2)^{\frac{p-2}{2}} + (p-2)((u^+)^2+\delta^2)^{\frac{p-4}{2}}(u^+)^2$$

for $p < 2$. In the case $p < 2$, we are allowed to estimate

$$u_+ f'_\delta = f_\delta\left[1 + (p-2)((u^+)^2+\delta^2)^{-1}(u^+)^2\right] \geq (1+p-2)f_\delta = (p-1)f_\delta,$$

whence $\infty > f'_\delta \geq (p-1)\frac{f_\delta}{u^+}$. If $p \geq 2$, it is clear that $f'_\delta = (p-1)\frac{f_\delta}{u^+}$. Using the Cauchy-Schwarz inequality for symmetric positive semidefinite matrices and Young's inequality, we conclude

$$\begin{aligned} I_1 &\geq -2\int_{\mathbb{R}^N} \sqrt{\langle A\nabla\eta_k, \nabla\eta_k\rangle}\sqrt{\frac{f_\delta u^+}{p-1}}\sqrt{\langle A\nabla u^+, u^+\rangle}\sqrt{f'_\delta}\eta_k \mathrm{d}x + \int_{\mathbb{R}^N} \left\langle A\nabla u^+, \nabla u^+ \right\rangle f'_\delta \eta_k^2 \mathrm{d}x \\ &\geq -\int_{\mathbb{R}^N} \frac{1}{p-1}\langle A\nabla\eta_k, \nabla\eta_k\rangle u^+ f_\delta + \left\langle A\nabla u^+, \nabla u^+ \right\rangle f'_\delta \eta_k^2 \mathrm{d}x + \int_{\mathbb{R}^N} \left\langle A\nabla u^+, \nabla u^+ \right\rangle f'_\delta \eta_k^2 \mathrm{d}x \\ &= -\frac{1}{p-1}\int_{\mathbb{R}^N} \langle A\nabla\eta_k, \nabla\eta_k\rangle u^+ f_\delta \mathrm{d}x. \end{aligned}$$

The estimate for the second integral is more involved. For $p < 2$ we calculate by partial integration

$$I_2 = -\int_{\mathbb{R}^N} \operatorname{div}(b)\eta_k^2(u^+)^2\left((u^+)^2+\delta^2\right)^{\frac{p-2}{2}}\mathrm{d}x - \int_{\mathbb{R}^N}\left\langle \nabla\eta_k^2, b\right\rangle(u^+)^2\left((u^+)^2+\delta^2\right)^{\frac{p-2}{2}}\mathrm{d}x$$
$$-\int_{\mathbb{R}^N}\left\langle \nabla u^+, b\right\rangle\eta_k^2\left[u^+\left((u^+)^2+\delta^2\right)^{\frac{p-2}{2}}+(p-2)\left((u^+)^2+\delta^2\right)^{\frac{p-4}{2}}(u^+)^3\right]\mathrm{d}x.$$

Next, we add the last term to both sides of the equation and obtain

$$\int_{\mathbb{R}^N}\left\langle \nabla u^+, b\right\rangle\eta_k^2\left[2u^+\left((u^+)^2+\delta^2\right)^{\frac{p-2}{2}}+(p-2)\left((u^+)^2+\delta^2\right)^{\frac{p-4}{2}}(u^+)^3\right]\mathrm{d}x$$
$$= -\int_{\mathbb{R}^N} \operatorname{div}(b)\eta_k^2(u^+)^2\left((u^+)^2+\delta^2\right)^{\frac{p-2}{2}} - \int_{\mathbb{R}^N}\left\langle \nabla\eta_k^2, b\right\rangle(u^+)^2\left((u^+)^2+\delta^2\right)^{\frac{p-2}{2}}\mathrm{d}x$$

so that, as $\delta\to 0$, we, at least formally, end up with

$$\begin{aligned}\lim_{\delta\to 0} I_2 &= \int_{\mathbb{R}^N}\left\langle \nabla u^+, b\right\rangle(u^+)^{p-1}\eta_k^2\mathrm{d}x\\ &= -\frac{1}{p}\int_{\mathbb{R}^N}\operatorname{div}(b)(u^+)^p\eta_k^2\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N}\left\langle \nabla\eta_k^2, b\right\rangle(u^+)^p\mathrm{d}x\\ &\geq -\frac{M}{p}\int_{\mathbb{R}^N}(u^+)^p\eta_k^2\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N}\left\langle \nabla\eta_k^2, b\right\rangle(u^+)^p\mathrm{d}x.\end{aligned}$$

If $p\geq 2$, one can perform essentially the same calculations but does not have to consider the limit $\delta\to 0$. The last integral I_3 is estimated from below as

$$I_3 \geq -M\left\langle u, \eta_k^2 f_\delta\right\rangle,$$

since $uu^+\geq 0$.

Let us justify why we are allowed to take the limit $\delta\to 0$ in the respective inequalities. For fixed $k\in\mathbb{N}$ every integrand appearing has compact support given by the support of η_k. Furthermore, it holds the monotone convergence of $f_\delta\to(u^+)^{p-1}$ and of $((u^+)^2+\delta^2)^{\frac{p}{2}}\to(u^+)^p$. Thus, by the theorem of dominated convergence, we conclude

$$\begin{aligned}\left\langle Eu, \eta_k^2(u^+)^{p-1}\right\rangle \geq &-\frac{1}{p-1}\int_{\mathbb{R}^N}\left\langle A\nabla\eta_k, \nabla\eta_k\right\rangle(u^+)^p\mathrm{d}x\\ &-\frac{M}{p}\int_{\mathbb{R}^N}(u^+)^p\eta_k^2\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N}\left\langle \nabla\eta_k^2, b\right\rangle(u^+)^p\mathrm{d}x\\ &- M\left\langle u, \eta_k^2(u^+)^{p-1}\right\rangle.\end{aligned}$$

It remains to investigate the limit $k\to\infty$. We note that $\eta_k(x)\to 1$ pointwise boundedly as $k\to\infty$. Moreover, it holds $\nabla\eta_k(x) = \frac{1}{k}[\nabla\eta]\left(\frac{1}{k}x\right)$ so that $\nabla\eta_k\to 0$ uniformly on $\mathbb{R}^N$.

From $u \in L^p(\mathbb{R}^N)$, it follows that $(u^+)^{p-1}$ is an element of the dual space $L^q(\mathbb{R}^N)$ of $L^p(\mathbb{R}^N)$. Using the theorem of dominated convergence and the duality of $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$, we deduce

$$\left\langle Eu, \eta_k^2(u^+)^{p-1}\right\rangle \to \left\langle Au, (u^+)^{p-1}\right\rangle \text{ and } \left\langle u, \eta_k^2(u^+)^{p-1}\right\rangle \to \left\langle u, (u^+)^{p-1}\right\rangle$$

for $k \to \infty$. We see that we need to prove that the remaining two integrals converge to zero. We recall that A grows at most of order $|x|^2$. The derivative of the cutoff function η_k is bounded by a constant multiplied by $\frac{1}{k}$ and its support is contained in $B_{3k}(0)$. This shows that $\langle A\nabla\eta_k, \nabla\eta_k\rangle$ is bounded independently of $k \in \mathbb{N}$. By the dominated convergence theorem, we deduce

$$\int_{\mathbb{R}^N} \langle A\nabla\eta_k, \nabla\eta_k\rangle\, (u^+)^p \mathrm{d}x \to 0$$

as $k \to \infty$. Similarly, due to the at most linear growth of b we see that $\langle b, \nabla\eta_k\rangle$ is bounded uniformly in k and therefore

$$2\int_{\mathbb{R}^N} \langle b, \nabla\eta_k\rangle\, \eta_k (u^+)^p \mathrm{d}x \to 0.$$

We conclude

$$\left\langle Eu, (u^+)^{p-1}\right\rangle \geq -\frac{M}{p}\int_{\mathbb{R}^N} (u^+)^p \mathrm{d}x - M\left\langle u, (u^+)^{p-1}\right\rangle \geq -2M\left\langle u, (u^+)^{p-1}\right\rangle.$$

Consequently, it holds

$$\left\langle (-E-2M)u, (u^+)^{p-1}\right\rangle \leq 0$$

for all $u \in C_c^\infty(\mathbb{R}^N)$. Recalling that for every $0 \neq u \in C_c^\infty(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ it holds

$$\left\|u^+\right\|_{p,\mathbb{R}^N}^{-\frac{p}{q}} (u^+)^{p-1} \in J(u) \subset L^q(\mathbb{R}^N),$$

we deduce the quasi-dispersiveness of the negative minimal realization $(-E, C_c^\infty(\mathbb{R}^N))$ by proposition A.1.6. □

Corollary 2.2.2. The minimal realization $(E, C_c^\infty(\mathbb{R}^N))$ is quasi-accretive.

Proof. This is the direct consequence of lemma A.1.10 and the previous result. □

Proposition 2.2.3. The minimal realization $(E, C_c^\infty(\mathbb{R}^N))$ is closable in $L^p(\mathbb{R}^N)$, the maximal realization $(\mathcal{E}_p, D(\mathcal{E}_p))$ is closed and it holds $\mathcal{R}(\lambda+\mathcal{E}_p) = L^p(\mathbb{R}^N)$ as well as $\mathcal{N}(\lambda+\overline{E}^p) = \{0\}$ for all $\lambda > 2M = 2\max\{b^\infty, c^\infty\}$.

Proof. We want to calculate the adjoint operator $(E', D(E'))$ of $(E, C_c^\infty(\mathbb{R}^N))$ in $L^p(\mathbb{R}^N)$. Since $C_c^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$, the adjoint operator is well-defined. As usual, we identify the dual of $L^p(\mathbb{R}^N)$ with $L^q(\mathbb{R}^N)$ so that the adjoint operator $(E', D(E'))$ of $(E, C_c^\infty(\mathbb{R}^N))$ is an operator acting on functions in $L^q(\mathbb{R}^N)$. Let $\psi \in D(E')$, then $\langle (E^T)^T\varphi, \psi\rangle = \langle E\varphi, \psi\rangle = \langle \varphi, E'\psi\rangle$ for all $\varphi \in C_c^\infty(\mathbb{R}^N)$ and $E'\psi \in L^q(\mathbb{R}^N)$, whence $\psi \in D(\mathcal{E}_q^T)$ and $\mathcal{E}_q^T\psi = E'\psi$. If conversely $\psi \in D(\mathcal{E}_q^T)$ and $\varphi \in C_c^\infty(\mathbb{R}^N) = D(E)$, then $\langle E\varphi, \psi\rangle = \langle \varphi, \mathcal{E}_q^T\psi\rangle$ and thus $\psi \in D(E')$ and $E'\psi = \mathcal{E}_q^T\psi$, hence $(E', D(E')) = (\mathcal{E}_q^T, D(\mathcal{E}_q^T))$. We deduce $E' = E^T$ regarded as differential operators on $C_c^\infty(\mathbb{R}^N)$.

As noted in the first part of this section, E^T satisfies the assumptions (A1) and (A2) as well. Consequently, the operator $(\mathcal{E}_p, D(\mathcal{E}_p)) = ((E^T, C_c^\infty(\mathbb{R}^N))'$ is closed, since the adjoint operator of a densely defined operator is always closed. Furthermore, $(E, C_c^\infty(\mathbb{R}^N))$ is closable with $\overline{(E, C_c^\infty(\mathbb{R}^N))}^p = ((E, C_c^\infty(\mathbb{R}^N))')' = (\mathcal{E}_q^T, D(\mathcal{E}_q^T))'$. The quasi-accretiveness of E^T as an operator in $L^q(\mathbb{R}^N)$, proven in corollary 2.2.2, shows that for $\lambda > 0$ sufficiently large it holds $\mathcal{N}(\lambda + E^T) = \{0\}$. Here E^T denotes the minimal realization $(E^T, C_c^\infty(\mathbb{R}^N))$. The closed range theorem implies that $\mathcal{R}(\lambda + \mathcal{E}_p) = \mathcal{N}\left(\lambda + E^T\right)^\perp = \{0\}^\perp = L^p(\mathbb{R}^N)$, since $\mathcal{R}(\lambda + \overline{E^T}^q) = \mathcal{R}(\lambda + \mathcal{E}_p')$ is closed. The closedness of $\mathcal{R}(\lambda + \overline{E^T}^q)$ is a direct consequence of the quasi-accretiveness of the closure $\overline{E^T}^q$ of the accretive operator E^T. We refer to lemma A.1.12 for further details. □

It remains to show that $\mathcal{R}(\lambda + \overline{E}^p) = L^p(\mathbb{R}^N)$. From this we would be able to deduce $\overline{(E, C_c^\infty(\mathbb{R}^N))}^p = (\mathcal{E}_p, D(\mathcal{E}_p))$ as well as the bijectivity of $\lambda + \mathcal{E}_p$. It turns out that it is easier to prove this result for an elliptic approximation E_ε of E first. This will be done in the next subsection.

2.2.3 Elliptic regularization

As mentioned before, there are two peculiarities concerning the differential operator E. The degenerate diffusion matrix A and the unbounded coefficients A and b. A well-known technique to deal with unbounded coefficients is smoothing and cutoff. We are going to combine this method with so-called elliptic regularization to treat our problem. Keeping this in mind, let us introduce the smooth elliptic regularization E_ε of E chosen as

$$E_\varepsilon u = -\mathrm{div}(A^\varepsilon \nabla u) + \langle b^\varepsilon, \nabla u\rangle + c^\varepsilon u$$

with corresponding coefficients

$$\begin{aligned} a_{ij}^\varepsilon &= \eta_\varepsilon(\omega_\varepsilon * a_{ij}) + \varepsilon\delta_{ij}, \\ b_i^\varepsilon &= \eta_\varepsilon(\omega_\varepsilon * b_j), \\ c^\varepsilon &= \eta_\varepsilon(\omega_\varepsilon * c) \end{aligned}$$

for $i, j = 1, \dots, N$ and $\varepsilon > 0$. The functions $\eta_\varepsilon, \omega_\varepsilon$ denote the cutoff function and the mollifier as defined in section A.3. The parameter ε is chosen as any sequence converging to 0.

Lemma 2.2.4. The elliptic regularization E_ε has smooth and bounded coefficients and is uniformly elliptic with constant $\varepsilon > 0$.

Proof. The smoothness and boundedness of the coefficients are a consequence of the convolution with the smoothing kernel and the cutoff. Further, it holds

$$\begin{aligned}\langle A^\varepsilon(x)\xi, \xi\rangle &= \sum_{i,j=1}^N \eta_k(x)(\omega_k * a_{ij})(x)\xi_i\xi_j + \varepsilon \sum_{i=1}^N \xi_i^2 \\ &= \eta_k(x) \int_{\mathbb{R}^N} \omega_k(x-y) \sum_{i,j=1}^N a_{ij}(y)\xi_i\xi_j \mathrm{d}y + \varepsilon \left|\xi\right|^2 \geq \varepsilon \left|\xi\right|^2,\end{aligned}$$

for all $x, \xi \in \mathbb{R}^N$, whence E_ε is uniformly elliptic with constant $\varepsilon > 0$. □

The coefficients of E_ε satisfy the assumptions (A1) and (A2) as well, so that the minimal and the maximal realization of E_ε are well-defined. In particular, the already obtained results also hold for the respective realizations of the differential operator E_ε. A big advantage of E_ε is that we may apply classical elliptic regularity theory. For the readers convenience we recall a partial result here.

Theorem 2.2.5. *Let E be a second order strictly elliptic differential operator as in equation 2.2.1 with smooth and bounded coefficients $A, b, c \in C^\infty(\mathbb{R}^N)$. Let $p \in (1, \infty)$. Assume that $u \in L^p(\mathbb{R}^N)$ is a distributional solution to the partial differential equation $Eu = g$, where $g \in C_c^\infty(\mathbb{R}^N)$, then $u \in C^\infty \cap D(\mathcal{E}_2)$. If $p = 2$, it holds $u \in W^{2,2}(\mathbb{R}^N)$. In particular, u is a classical solution of $Eu = g$.*

Proof. Let $p \in (1, \infty)$. Every elliptic partial differential operator of second order with smooth and bounded coefficients is hypoelliptic. In particular, we deduce that $u \in C^\infty(\mathbb{R}^N)$. For more information on this matter we refer to remark 2.4.3.

If $p = 2$, it holds $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N)$ and u is also a weak solution to $Eu = g$. Therefore, we can apply the interior regularity estimates from [Eva10, Section 6.3, Theorem 1] to $V(R) := B_{R+1}(0) \setminus B_R(0) \subset B_{R+2}(0) \setminus B_{R-1}(0) =: U(R)$ for every $R \in \mathbb{N}$. This shows

$$\|u\|_{2,2,V(R)} \leq C \left(\|f\|_{2,U(R)} + \|u\|_{2,U(R)} \right).$$

Summation of these estimates for all $R \in \mathbb{N}$ gives

$$\|u\|_{2,2,\mathbb{R}^N} \leq 3C \left(\|f\|_{2,\mathbb{R}^N} + \|u\|_{2,\mathbb{R}^N} \right) < \infty.$$ □

Lemma 2.2.6. The operator $(-E_\varepsilon, C_c^\infty(\mathbb{R}^N))$ is essentially quasi m-dispersive in $L^p(\mathbb{R}^N)$ uniformly in $\varepsilon \in (0,1)$, i.e. there is a constant $\lambda_0 > 0$ such that the dispersiveness property is satisfied for all $\lambda > \lambda_0$.

Proof. As explained, we are able to apply corollary 2.2.2 to deduce that $(-E_\varepsilon, C_c^\infty(\mathbb{R}^N))$ is quasi-dispersive in $L^p(\mathbb{R}^N)$. By definition A.1.5 and by lemma A.1.12, it remains to show that there is $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ the range $\mathcal{R}(\lambda + E_\varepsilon)$ is a dense subset of $L^p(\mathbb{R}^N)$. We suppose there is a function $u \in L^q(\mathbb{R}^N)$ such that

$$\langle u, (\lambda + E_\varepsilon)\varphi \rangle = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R}^N)$. In other words, there is a distributional solution $u \in L^q(\mathbb{R}^N)$ of the equation $(\lambda + E_\varepsilon^T)u = 0$. The elliptic regularity theory, i.e. theorem 2.2.5, implies that $u \in C^\infty \cap D(\mathcal{E}_{\varepsilon,q}^T) \subset L^q(\mathbb{R}^N)$ and in particular that u is a classical solution. We consider the function $(u^+)^{q-1} \in L^p(\mathbb{R}^N)$ and choose any approximating sequence $(\varphi_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$. Since $E_\varepsilon^T u \in L^q(\mathbb{R}^N)$, it holds

$$\left\langle (\lambda + E_\varepsilon^T)u, (u^+)^{q-1} \right\rangle = \lim_{k\to\infty} \left\langle (\lambda + E_\varepsilon^T)u, \varphi_k \right\rangle = \lim_{k\to\infty} \left\langle u, (\lambda + E_\varepsilon^T)\varphi_k \right\rangle = 0, \tag{2.2.3}$$

since $(\lambda + E_\varepsilon^T)\varphi_k$ is an admissible test function for all $k \in \mathbb{N}$. We want to apply proposition 2.2.1 and note that

$$M_\varepsilon = \max\{b^{\varepsilon,\infty}, c^{\varepsilon,\infty}\} \leq \max\{\varepsilon \max_{|\varepsilon x| \leq 2} |\langle [\nabla\eta](\varepsilon x), b(x)\rangle| + b^\infty, c^\infty\} \leq C + M \tag{2.2.4}$$

where $C > 0$ is a constant independent of $\varepsilon \in (0,1)$. On the one hand, we conclude, by proposition 2.2.1 applied to $u \in C^\infty \cap D(\mathcal{E}_{\varepsilon,q}^T)$, that

$$\left\langle E_\varepsilon^T u, (u^+)^{q-1} \right\rangle \geq -2M_\varepsilon \left\langle u, (u^+)^{q-1} \right\rangle \geq -2(C+M) \left\langle u, (u^+)^{q-1} \right\rangle$$

On the other hand, since u is a distributional solution and as seen in equation (2.2.3) the function $(u^+)^{q-1}$ is an admissible test function, it holds

$$\lambda \left\| u^+ \right\|_{q,\mathbb{R}^N} = -\left\langle E_\varepsilon^T u, (u^+)^{q-1} \right\rangle \leq 2(C+M) \left\| u^+ \right\|_{q,\mathbb{R}^N}.$$

If $\lambda > 2(C+M)$, this implies $u^+ = 0$. Since $-u$ is a weak solution of the equation $(\lambda + E_\varepsilon^T)w = 0$, too, we deduce $u^- = (-u)^+ = 0$. This shows that u must be zero. By duality of $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$, we conclude that $\mathcal{R}(\lambda + E_\varepsilon)$ is dense in $L^p(\mathbb{R}^N)$ for every $\lambda > 2(C + M) =: \lambda_0$. Finally, since $C_c^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$, we conclude that $(-E_\varepsilon, C_c^\infty(\mathbb{R}^N))$ is essentially quasi-m-dispersive. The uniformity of this statement in $\varepsilon \in (0,1)$ follows from the fact that the constant λ_0 was chosen independently of ε. $\square$

Corollary 2.2.7. The minimal realization $(E_\varepsilon, C_c^\infty(\mathbb{R}^N))$ is essentially quasi-m-accretive uniformly in ε. Moreover, it holds $\overline{(E_\varepsilon, C_c^\infty(\mathbb{R}^N))}^p = (\mathcal{E}_{\varepsilon,p}, D(\mathcal{E}_{\varepsilon,p}))$. In particular, the operator $(\mathcal{E}_{\varepsilon,p}, D(\mathcal{E}_{\varepsilon,p}))$ is quasi-m-accretive.

Proof. We apply lemma 2.2.6 to deduce the quasi-m-accretiveness of $(E_\varepsilon, C_c^\infty(\mathbb{R}^N))$ from lemma A.1.10. The operator $(\mathcal{E}_{\varepsilon,p}, D(\mathcal{E}_{\varepsilon,p}))$ is quasi-accretive, since it is an elliptic second order differential operator with bounded and smooth coefficients. This is for example proven in [CV88, Theorem 5.2]. Furthermore, by proposition 2.2.3, the operator $(\mathcal{E}_{\varepsilon,p}, D(\mathcal{E}_{\varepsilon,p}))$ is closed and hence it holds $\overline{(E_\varepsilon, C_c^\infty(\mathbb{R}^N))}^p \subset (\mathcal{E}_{\varepsilon,p}, D(\mathcal{E}_{\varepsilon,p}))$. Let $u \in D(\mathcal{E}_{\varepsilon,p})$, then $\lambda u + \mathcal{E}_{\varepsilon,p}u \in L^p(\mathbb{R}^N)$ for every $\lambda > 0$. By the quasi-m-accretiveness of $\overline{E_\varepsilon}^p$, there is v in the domain of $\overline{E_\varepsilon}^p$ such that $\lambda u + \mathcal{E}_{\varepsilon,p}u = \lambda v + \overline{E_\varepsilon}^p v = \lambda v + \mathcal{E}_{\varepsilon,p}v$ if λ is chosen large enough. This implies that $u = v$ by the quasi-accretiveness of $\mathcal{E}_{\varepsilon,p}$ for λ large enough. □

Proposition 2.2.8. Let $g \in C_c^\infty(\mathbb{R}^N)$. We assume that $u \in L^2(\mathbb{R}^N)$ is a distributional solution of the equation $(\lambda + \mathcal{E}_{\varepsilon,2})u = g$ for some $\lambda > 0$. In this case it holds $u \in W^{1,\infty}(\mathbb{R}^N)$.

Let us comment on this result. It is clear that, by elliptic regularity, the function u and its derivatives are bounded on every compact set so that it remains to provide a bound in infinity. We recall that in infinity we basically have $E_\varepsilon = \varepsilon\Delta$. We are going to prove the boundedness in infinity using a statement which is due to Jürgen Moser concerning the local boundedness of weak subsolutions. Due to the fact that in infinity we are in the case of constant coefficients, we may differentiate the equation and apply this result to gradient, too. This method is often called a $L^2 - L^\infty$ bound because the bound will depend on the L^2 norm of the weak solution u.

Theorem 2.2.9. *Let $x \in \mathbb{R}^N$, $r > 0$ and $A \in L^\infty(\mathbb{R}^N; \mathbb{R}^{N\times N})$ be uniformly elliptic with constant $\lambda > 0$. We assume that $0 \le u \in W^{1,2}(B_{2r}(y))$ is a weak subsolution of*

$$\operatorname{div}(A\nabla u) = 0$$

in $B_{2r}(x)$, i.e. it holds

$$\int_{B_{2r}(x)} \langle A\nabla u, \nabla\varphi\rangle \le 0$$

for all $0 \le \varphi \in C_c^\infty(B_{2r}(x))$. Then there exists a constant $c = c(N, \lambda, \|A\|_{\infty,\mathbb{R}^N})$ such that

$$\operatorname*{ess\,sup}_{B_r(x)} u^2(x) \le \frac{c}{r^N} \int_{B_{2r}(x)} u^2 \mathrm{d}x.$$

Proof. [Mos60, Theorem 1] □

Proof of proposition 2.2.8. As seen in the proof of lemma 2.2.6, by elliptic regularity, it is $u \in C^\infty \cap W^{2,2}(\mathbb{R}^N)$. Let $R > \frac{1}{3\varepsilon}$ such that $\operatorname{supp} g \subset B_R(0)$. In particular, it holds

$$\operatorname{supp}(\eta_\varepsilon(\omega_\varepsilon * a_{ij})), \operatorname{supp}(\eta_\varepsilon(\omega_\varepsilon * b_j)), \operatorname{supp}(\eta_\varepsilon(\omega_\varepsilon * c)) \subset B_R(0)$$

for all $i, j = 1, \ldots, N$. Let $r > 0$, then, by continuity of u and ∇u, we immediately deduce $u \in L^\infty(\overline{B_{R+2r}(0)})$ and that $\nabla u \in L^\infty(\overline{B_{R+2r}(0)}, \mathbb{R}^N)$. Thus, it remains to prove the boundedness outside of $B_{R+2r}(0)$. This will be done by using theorem 2.2.9. Let $y \in \overline{B_{R+2r}(0)}^c$. We note that due to the choice of R the equation $(\lambda + E_\varepsilon)u = g$ in $B_{2r}(y)$ reduces to $\varepsilon \Delta u = -\lambda u$ in $B_{2r}(y)$. We want to show that the absolute value of u is a weak subsolution of the equation $\Delta w = 0$ in $B_{2r}(y)$. We approximate $|u|$ by $\sqrt{u^2 + \delta^2}$ for $\delta > 0$. Since $u \in W^{1,2}(B_{2r}(y))$, it holds $\sqrt{u^2 + \delta^2} \to |u|$ in $W^{1,2}(B_{2r}(y))$. For $0 \leq \varphi \in C_c^\infty(\mathbb{R}^N)$, it is

$$\begin{aligned}
\int_{B_{2r}(y)} \left\langle \nabla \sqrt{u^2 + \delta^2}, \nabla \varphi \right\rangle \mathrm{d}x &= \int_{B_{2r}(y)} \left\langle \nabla u, \nabla \left(\frac{\varphi u}{\sqrt{u^2 + \delta^2}} \right) \right\rangle \mathrm{d}x \\
&\quad - \int_{B_{2r}(y)} \left\langle \nabla u, \nabla \left(\frac{u}{\sqrt{u^2 + \delta^2}} \right) \right\rangle \varphi \mathrm{d}x \\
&= -\frac{\lambda}{\varepsilon} \int_{B_{2r}(y)} \frac{u^2}{\sqrt{u^2 + \delta^2}} \varphi \mathrm{d}x - \int_{B_{2r}(y)} \frac{|\nabla u|^2}{\sqrt{u^2 + \delta^2}} \varphi \mathrm{d}x \\
&\quad + \int_{B_{2r}(y)} \frac{|\nabla u|^2}{\sqrt{u^2 + \delta^2}} \frac{u^2}{u^2 + \delta^2} \varphi \mathrm{d}x \\
&\leq 0 - \int_{B_{2r}(y)} \frac{|\nabla u|^2}{\sqrt{u^2 + \delta^2}} \varphi \mathrm{d}x + \int_{B_{2r}(y)} \frac{|\nabla u|^2}{\sqrt{u^2 + \delta^2}} \varphi \mathrm{d}x = 0.
\end{aligned}$$

Consequently, $|u| \in W^{1,2}(B_{2r}(y))$ is a nonnegative weak subsolution of $\Delta w = 0$ in $B_{2r}(y)$. By theorem 2.2.9, we conclude

$$\operatorname*{ess\,sup}_{B_r(y)} u^2 \leq \frac{c}{r^N} \int_{B_{2r}(y)} u^2 \mathrm{d}x \leq \frac{c}{r^N} \|u\|_{2,\mathbb{R}^N}^2 < \infty.$$

In the ball $B_{2r}(y)$ the equation $(\lambda + \mathcal{E}_{\varepsilon,2})w = g$ in $B_{2r}(y)$ reduces to a partial differential equation with constant coefficients. Since $u \in C^\infty(\mathbb{R}^N)$, we can differentiate the equation to deduce that for every $k = 1, \ldots, N$ the function $\partial_k u$ is a solution in $B_{2r}(y)$, too. In particular, we can repeat our argument to deduce that $|\partial_{x_k} u|$ is a weak subsolution of the equation $\Delta w = 0$ in $B_{2r}(y)$. Therefore, by theorem 2.2.9, it follows

$$\operatorname*{ess\,sup}_{B_r(y)} |\partial_{x_k} u|^2 \leq \frac{c}{r^N} \int_{B_{2r}(y)} |\partial_{x_k} u|^2 \mathrm{d}x \leq \frac{c}{r^N} \|u\|_{1,2,\mathbb{R}^N}^2 < \infty.$$

This shows that u and ∇u are bounded on $\overline{B_{R+2r}(0)}^c$ due to the fact that $y \in \overline{B_{R+2r}(0)}^c$ can be chosen arbitrarily and the bound does not depend on the choice of y. Consequently, it holds $u \in W^{1,\infty}(\mathbb{R}^N)$. $\square$

2.2.4 Refined decay estimates for weak solutions of $\lambda u + E_\varepsilon u = g$

For $K \in \mathbb{N}$ we define $E_K u = (1+|x|^2)^{\frac{K}{2}} E (1+|x|^2)^{-\frac{K}{2}} u$ and $E_0 = E$. The differential operators E_K are once again of the type of equation (A.1.1) with coefficients

$$
\begin{aligned}
&a_{ij}^K(x) = a_{ij}(x),\\
&b_i^K(x) = b_i(x) + \sum_{j=1}^n \frac{2K a_{ij} x_j}{1+|x|^2},\\
&c^K(x) = c(x) + \frac{K\,\mathrm{tr}(A)}{1+|x|^2} - \frac{K(K+2)}{(1+|x|^2)^2}\langle Ax, x\rangle - \frac{K}{1+|x|^2}\langle b, x\rangle + \sum_{i,j=1}^N K x_j \frac{\partial_{x_j} a_{ij}}{1+|x|^2}.
\end{aligned}
$$

We note that the coefficients A_K, b_K and c_K satisfy the same conditions as the coefficients A, b, c. Hence, all results we have proven for E remain valid for E_K. We define $M_K = \max\{b^{K,\infty}, c^{K,\infty}\}$ and

$$E_{\varepsilon,K} = (E_\varepsilon)_K = (1+|x|^2)^{\frac{K}{2}} E_\varepsilon (1+|x|^2)^{-\frac{K}{2}}.$$

Lemma 2.2.10. Let $K > N$. There exists a constant $\lambda_1 > 2\max\{M_0, M_K, M_{2K}\}$ such that for every $\lambda > \lambda_1$ and any distributional solution $u \in L^p(\mathbb{R}^N)$ of the equation

$$(\lambda + E_\varepsilon) u = g,$$

where $g \in C_c^\infty(\mathbb{R}^N)$, it holds

$$(1+|x|^2)^{\frac{k}{2}} u \in W^{1,1} \cap W^{1,\infty}(\mathbb{R}^N) \subset W^{1,p}(\mathbb{R}^N)$$

for all $0 \le k \le K$.

Proof. We define the function $g_k = (1+|x|^2)^{\frac{k}{2}} g \in C_c^\infty(\mathbb{R}^N)$ for all $k \in \mathbb{N}$. Moreover, we choose λ_1 as the maximal constant of the constants λ_0 given by lemma 2.2.6 for each of the operators $\mathcal{E}_{\varepsilon,0,p}$, $\mathcal{E}_{\varepsilon,K,p}$ and $\mathcal{E}_{\varepsilon,2K,p}$. As seen in the proof of lemma 2.2.6, it holds $\lambda_1 > 2\max\{M_0, M_K, M_{2K}\}$. By corollary 2.2.7, the differential operators $(\mathcal{E}_{\varepsilon,0,p}, D(\mathcal{E}_{\varepsilon,0,p}))$, $(\mathcal{E}_{\varepsilon,K,p}, D(\mathcal{E}_{\varepsilon,K,p}))$ and $(\mathcal{E}_{\varepsilon,2K,p}, D(\mathcal{E}_{\varepsilon,2K,p}))$ are quasi-m-accretive for any constant $\lambda > \lambda_1$. Consequently, there exist unique distributional solutions $u_{\varepsilon,0,p}, u_{\varepsilon,K,p}, u_{\varepsilon,2K,p} \in L^p(\mathbb{R}^N)$ of

$$(\lambda + \mathcal{E}_{\varepsilon,0,p}) u_{\varepsilon,0,p} = g, \quad (\lambda + \mathcal{E}_{\varepsilon,K,p}) u_{\varepsilon,K,p} = g_K \text{ and } (\lambda + \mathcal{E}_{\varepsilon,2K,p}) u_{\varepsilon,2K,p} = g_{2K},$$

respectively. Multiplying the last equation by $(1+|x|^2)^{-\frac{K}{2}}$ from the left, it follows that

$$(\lambda+\mathcal{E}_{\varepsilon,K})(1+|x|^2)^{-\frac{K}{2}}u_{\varepsilon,2K,p}=g_K.$$

By uniqueness, i.e. the quasi-dispersiveness of $\mathcal{E}_{\varepsilon,K}$, we conclude that $(1+|x|^2)^{-\frac{K}{2}}u_{\varepsilon,2K,p}=u_{\varepsilon,K,p}$. The same argument shows that $u_{\varepsilon,K,p}=(1+|x|^2)^{\frac{K}{2}}u_{\varepsilon,0,p}$. To apply proposition 2.2.8, we need to choose $p=2$. By doing so, we deduce $u_{\varepsilon,K,2},u_{\varepsilon,2K,2}\in W^{1,\infty}(\mathbb{R}^N)$. It holds $(1+|x|^2)^{-\frac{K}{2}}\in W^{1,1}(\mathbb{R}^N)$ for $K>N$. Therefore,

$$u_{\varepsilon,K,2}=(1+|x|^2)^{-\frac{K}{2}}u_{\varepsilon,2K,2}\in W^{1,1}\cap W^{1,\infty}(\mathbb{R}^N)\subset L^p(\mathbb{R}^N).$$

The latter set relation follows by interpolation of the Lebesgue spaces. We deduce that $u_{\varepsilon,K,2}$ solves $(\lambda+E_{\varepsilon,K})u_{\varepsilon,K,2}=g_K$ in $L^p(\mathbb{R}^N)$ and therefore, by uniqueness, it must hold $u_{\varepsilon,K,p}=u_{\varepsilon,K,2}$ for every $p\in(1,\infty)$. Consequently, it is $u_{\varepsilon,K,p}\in W^{1,1}\cap W^{1,\infty}(\mathbb{R}^N)$ and since $(1+|x|^2)^{\frac{k}{2}}\le(1+|x|^2)^{\frac{K}{2}}$ for all $0\le k\le K$, we conclude $(1+|x|^2)^{\frac{k}{2}}u_{\varepsilon,0,p}=(1+|x|^2)^{\frac{k-K}{2}}u_{\varepsilon,K,p}\in W^{1,1}\cap W^{1,\infty}(\mathbb{R}^N)\subset W^{1,p}(\mathbb{R}^N)$. □

2.2.5 Towards a uniform bound for weak solutions of $\lambda u+E_\varepsilon u=g$

It is our aim to use the regularized operators E_ε to construct a solution to the equation $(\lambda+E)u=g$ for λ sufficiently large. To do so, we want to take the limit $\varepsilon\to 0$ of the solutions u_ε to $(\lambda+E_\varepsilon)u_\varepsilon=g$ and thus we need uniform bounds on $u_\varepsilon,\nabla u_\varepsilon$ independent of $\varepsilon>0$. The inequality of the next proposition provides us with the required tools to prove such bounds. The proof of this proposition will make use of the so-called Oleinik inequality.

Theorem 2.2.11 (Oleinik inequality). *Let $A\in C^2(\mathbb{R}^N;\mathbb{R}^{N\times N})$ with bounded second derivatives such that $A(x)$ is symmetric positive semidefinite for all $x\in\mathbb{R}^N$. For all matrices $Q\in\mathbb{R}^{N\times N}$ and every $i=1,\dots,N$ the inequality*

$$\operatorname{tr}([\partial_{x_i}A]Q)^2\le 4N^2A^\infty\operatorname{tr}(QAQ^T)$$

holds in $\mathbb{R}^N$. In particular, for all $u\in C^2(\mathbb{R}^N)$ and all $i=1,\dots,N$ it holds

$$\operatorname{tr}([\partial_{x_i}A]\nabla^2u)^2\le 4N^2A^\infty\operatorname{tr}(\nabla^2uA\nabla^2u).$$

This theorem is due to Olga Oleinik and her proof can be found [Ole73, Lemma 1.7]. The proof presented here is taken from [SV06, Lemma 3.2.3].

Proof. We start with an elementary estimate for a nonnegative function $\varphi\in C^2(\mathbb{R})$ whose second derivative is bounded by a constant $c\ge 0$. Let $x,y\in\mathbb{R}$, then, by Taylor expansion in

x evaluated at $x+y$, it holds

$$0 \leq \varphi(x+y) \leq \varphi(x) + \varphi'(x)y + \frac{c}{2}y^2.$$

Therefore, the quadratic polynomial $y \mapsto \varphi(x) + \varphi'(x)y + \frac{c}{2}y^2$ is nonnegative for all $y \in \mathbb{R}$. Consequently, it can either have a single real root or two imaginary roots. The quadratic formula implies that $(\varphi'(x))^2 - 2c\varphi(x) \leq 0$ and consequently

$$|\varphi'(x)| \leq \sqrt{2c\varphi(x)}.$$

Let $i \in \{1, \dots, N\}$ and fix $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in \mathbb{R}$. For $j, k = 1, \dots, N$ we define the real-valued function $\varphi_\pm^{j,k} \colon \mathbb{R} \to [0, \infty)$ by

$$\varphi_\pm^{j,k}(x) = \langle e_j \pm e_k, A(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)(e_j \pm e_k) \rangle .$$

Since the second derivative of $\varphi_\pm^{j,k}$ is bounded by $4A^\infty$, we conclude

$$\left| \partial_x \varphi_\pm^{j,k}(x) \right| \leq \sqrt{8A^\infty} \sqrt{\varphi_\pm^{j,k}(x)}$$

for all $x \in \mathbb{R}$. Writing $a_{jk} = \frac{1}{4}\left(\varphi_+^{j,k} - \varphi_-^{j,k}\right)$, we deduce

$$\begin{aligned}
|\partial_{x_i} a_{jk}(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)| &\leq \frac{1}{4}\left(\left|\partial_{x_i}\varphi_+^{j,k}\right|(x_i) + \left|\partial_{x_i}\varphi_-^{j,k}\right|(x_i) \right) \\
&\leq \frac{\sqrt{2A^\infty}}{2}\left(\sqrt{\left|\varphi_+^{j,k}\right|(x_i)} + \sqrt{\left|\varphi_-^{j,k}\right|(x_i)} \right) \\
&\leq \sqrt{2A^\infty}\sqrt{\varphi_+^{j,k}(x_i) + \varphi_-^{j,k}(x_i)} \\
&= \sqrt{2A^\infty}\sqrt{a_{jj}(x) + a_{kk}(x)}.
\end{aligned}$$

Let $x \in \mathbb{R}^N$. We consider the case that $A(x)$ is diagonal first. In this case it holds

$$\begin{aligned}
\operatorname{tr}([\partial_{x_i} A](x)Q)^2 &= \left(\sum_{j=1}^N \sum_{k=1}^N [\partial_{x_i} a_{jk}](x) Q_{kj} \right)^2 \leq N \sum_{j=1}^N \left(\sum_{k=1}^N [\partial_{x_i} a_{jk}](x) Q_{jk} \right)^2 \\
&\leq N^2 \sum_{j=1}^N \sum_{k=1}^N \left[\partial_{x_i} a_{jk}\right]^2(x) Q_{jk}^2 \leq 2N^2 A^\infty \sum_{j=1}^N \sum_{k=1}^N (a_{jj}(x) + a_{kk}(x)) Q_{jk}^2 \\
&\leq 4N^2 A^\infty \sum_{j,k=1}^N Q_{jk} a_{kk}(x) Q_{kj}^T = 4N^2 A^\infty \operatorname{tr}(QA(x)Q^T)
\end{aligned}$$

for any matrix Q. If $A(x)$ is not diagonal, choose an orthogonal basis S such that $S^TA(x)S$ is diagonal and then apply above inequality to $S^TA(x)S$ and the matrix S^TQS, then

$$\begin{aligned}\operatorname{tr}([\partial_{x_i}A](x)Q)^2 &= \operatorname{tr}(S^T[\partial_{x_i}A](x)SS^TQS)^2 = \operatorname{tr}([\partial_{x_i}S^TAS](x)S^TQS)^2\\ &\leq 4N^2A^\infty \operatorname{tr}(S^TQSS^TA(x)SS^TQ^TS) = 4N^2A^\infty \operatorname{tr}(QA(x)Q^T).\end{aligned}$$

This shows the theorem because $x \in \mathbb{R}^N$ can be chosen arbitrarily. □

Proposition 2.2.12. We assume that the zeroth order term of the differential operator E satisfies $c \in C_b^1(\mathbb{R}^N)$ with bounded first derivatives. Let $u \in W^{1,p} \cap C^2(\mathbb{R}^N) \cap D(\mathcal{E}_p)$ such that $\nabla(Eu) \in L^p(\mathbb{R}^N)$. Under this additional assumption there exists a constant $\lambda_2 > 0$ depending on the dimension, on the second derivatives of A, the first derivatives of b and c and on the bound of c such that the inequality

$$\left\langle Eu, |u|^{p-2}u\right\rangle + \left\langle \nabla(Eu), |\nabla u|^{p-2}\nabla u\right\rangle \geq -\lambda_2 \|u\|_{1,p,\mathbb{R}^N}^p$$

holds.

Proof. We set

$$M = \max\{A^\infty, b^\infty, c^\infty, \|\nabla c\|_{\infty,\mathbb{R}^N}\}. \tag{2.2.5}$$

Formally, we can integrate by parts on the left-hand side in the claimed inequality to obtain

$$\left\langle Eu, -\operatorname{div}(|\nabla u|^{p-2}\nabla u)\right\rangle = \int_{\mathbb{R}^N} \left[-\operatorname{div}(A\nabla u) + \langle b, \nabla u\rangle + cu\right](-\operatorname{div}(|\nabla u|^{p-2}\nabla u))\mathrm{d}x.$$

This will be the starting point of our proof. However, this integral does not need to be well-defined. Therefore, as in the proof of proposition 2.2.1, let us introduce the cutoff functions η_k and approximate $|\nabla u|^{p-2}$ as $h_\delta(|\nabla u|) = (|\nabla u|^2 + \delta^2)^{\frac{p-2}{2}}$ for $p < 2$ with the convention that $h_\delta(|\nabla u|) = |\nabla u|^{p-2}$ if $p \geq 2$. We further introduce the function $H_\delta(\nabla u) = h_\delta(|\nabla u|)\nabla u$. We want to estimate

$$\begin{aligned}\left\langle Eu, \eta_k^2(-\operatorname{div}(H_\delta(\nabla u)))\right\rangle &= \int_{\mathbb{R}^N} \left[\operatorname{div}(A\nabla u) - \langle b, \nabla u\rangle - cu\right]\operatorname{div}(H_\delta(\nabla u))\eta_k^2\mathrm{d}x\\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Let us investigate each of the integrals I_1, I_2, I_3 separately. We perform the estimation of the integral I_1 first. It holds

$$I_1 = \int_{\mathbb{R}^N} \operatorname{div}(A\nabla u)\operatorname{div}(H_\delta(\nabla u))\eta_k^2\mathrm{d}x = \sum_{i,j,l=1}^N \int_{\mathbb{R}^N} \partial_{x_i}(a_{ij}(\partial_{x_j}u))\partial_{x_l}((\partial_{x_l}u)h_\delta)\eta_k^2\mathrm{d}x$$

$$
\begin{aligned}
&= \sum_{i,j,l=1}^{N} \int_{\mathbb{R}^N} -\partial_{x_l}\left(\partial_{x_i}(a_{ij}(\partial_{x_j} u))\eta_k^2\right)((\partial_{x_l} u)h_\delta)\mathrm{d}x \\
&= \sum_{i,j,l=1}^{N} \int_{\mathbb{R}^N} -(\partial_{x_i}\partial_{x_l}(a_{ij}(\partial_{x_j} u)))(\partial_{x_l} u)h_\delta\eta_k^2 - \left(\partial_{x_i}(a_{ij}\partial_{x_j} u)\right)(\partial_{x_l} u)(\partial_{x_l}\eta_k^2)h_\delta\mathrm{d}x \\
&= \sum_{i,j,l=1}^{N} \int_{\mathbb{R}^N} -\partial_{x_i}((\partial_{x_l} a_{ij})(\partial_{x_j} u))(\partial_{x_l} u)h_\delta\eta_k^2 - \partial_{x_i}(a_{ij}(\partial_{x_l}\partial_{x_j} u))(\partial_{x_l} u)h_\delta\eta_k^2 \\
&\qquad - \left(\partial_{x_i}(a_{ij}(\partial_{x_j} u))\right)(\partial_{x_l} u)(\partial_{x_l}\eta_k^2)h_\delta\mathrm{d}x \\
&= \sum_{i,j,l=1}^{N} \int_{\mathbb{R}^N} -\partial_{x_i}((\partial_{x_l} a_{ij})(\partial_{x_j} u))(\partial_{x_l} u)h_\delta\eta_k^2 + a_{ij}(\partial_{x_l}\partial_{x_j} u)(\partial_{x_l} u)(\partial_{x_i}\eta_k^2)h_\delta \\
&\qquad a_{ij}(\partial_{x_l}\partial_{x_j} u)\partial_{x_i}((\partial_{x_l} u)h_\delta)\eta_k^2 - \left(\partial_{x_i}(a_{ij}(\partial_{x_j} u))\right)(\partial_{x_l} u)(\partial_{x_l}\eta_k^2)h_\delta\mathrm{d}x \\
&= \sum_{i,j,l=1}^{N} \int_{\mathbb{R}^N} -\partial_{x_i}((\partial_{x_l} a_{ij})(\partial_{x_j} u))(\partial_{x_l} u)h_\delta\eta_k^2 + a_{ij}(\partial_{x_l}\partial_{x_j} u)(\partial_{x_l} u)(\partial_{x_i}\eta_k^2)h_\delta \\
&\qquad a_{ij}(\partial_{x_l}\partial_{x_j} u)\partial_{x_i}((\partial_{x_l} u)h_\delta)\eta_k^2 + (a_{ij}\partial_{x_j} u)(\partial_{x_l} u)(\partial_{x_i}\partial_{x_l}\eta_k^2)h_\delta \\
&\qquad + (a_{ij}(\partial_{x_j} u))\partial_{x_i}((\partial_{x_l} u)h_\delta)(\partial_{x_l}\eta_k^2)\mathrm{d}x \\
&= \sum_{i,j,l=1}^{N} \int_{\mathbb{R}^N} -\partial_{x_i}((\partial_{x_l} a_{ij})(\partial_{x_j} u))(\partial_{x_l} u)h_\delta\eta_k^2 + a_{ij}(\partial_{x_l}\partial_{x_j} u)(\partial_{x_l} u)(\partial_{x_i}\eta_k^2)h_\delta \\
&\qquad a_{ij}(\partial_{x_l}\partial_{x_j} u)(\partial_{x_i}((\partial_{x_l} u)h_\delta))\eta_k^2 + a_{ij}(\partial_{x_j} u)(\partial_{x_l} u)(\partial_{x_i}\partial_{x_l}\eta_k^2)h_\delta \\
&\qquad a_{ij}(\partial_{x_j} u)(\partial_{x_l} u)(\partial_{x_i} h_\delta)(\partial_{x_l}\eta_k^2) + a_{ij}(\partial_{x_j} u)(\partial_{x_l}\eta_k^2)(\partial_{x_i}\partial_{x_l} u)h_\delta\mathrm{d}x \\
&=: X_1 + X_2 + X_3 + \mathcal{O}_\delta^1 + X_4 + X_2.
\end{aligned}
$$

Here we have used that the second and fifth term are the same due to the symmetry of A and $\nabla^2 u$. To estimate the second integral I_2, we note that $-|\nabla u|^2 \geq -(|\nabla u|^2 + \delta^2)$, whence

$$
-|\nabla u|^2 h_\delta \geq -\left(|\nabla u|^2 + \delta^2\right)^{\frac{p}{2}} \tag{2.2.6}
$$

and we proceed as follows

$$
\begin{aligned}
I_2 &= -\int_{\mathbb{R}^N} \langle b, \nabla u\rangle \operatorname{div}(H_\delta(\nabla u))\eta_k^2 \mathrm{d}x = -\sum_{i,j=1}^N \int_{\mathbb{R}^N} b_i(\partial_{x_i} u)\partial_{x_j}((\partial_{x_j} u)h_\delta)\eta_k^2 \mathrm{d}x \\
&= \sum_{i,j=1}^N \int_{\mathbb{R}^N} \partial_{x_j}(b_i(\partial_{x_i} u)\eta_k^2)(\partial_{x_j} u)h_\delta \mathrm{d}x \\
&= \sum_{i,j=1}^N \int_{\mathbb{R}^N} (\partial_{x_j} b_i)(\partial_{x_i} u)(\partial_{x_j} u)h_\delta\eta_k^2 + (\partial_{x_j}\partial_{x_i} u)(\partial_{x_j} u)b_i h_\delta \eta_k^2 + (\partial_{x_i} u)(\partial_{x_j} u)(\partial_{x_j}\eta_k^2)b_i h_\delta \mathrm{d}x \\
&= \sum_{i,j=1}^N \int_{\mathbb{R}^N} (\partial_{x_j} b_i)(\partial_{x_i} u)(\partial_{x_j} u)h_\delta\eta_k^2 + b_i\partial_{x_i}\left(\frac{1}{p}h_\delta^{\frac{p}{p-2}}\right)\eta_k^2 + (\partial_{x_i} u)(\partial_{x_j} u)(\partial_{x_j}\eta_k^2)b_i h_\delta \mathrm{d}x \\
&=: \int_{\mathbb{R}^N} \langle [Db]\nabla u, \nabla u\rangle h_\delta \eta_k^2 + \left\langle b, \nabla\left(\frac{1}{p}(|\nabla u|^2+\delta^2)^{\frac{p}{2}}\right)\right\rangle \eta_k^2 \mathrm{d}x + \mathcal{O}_\delta^3 \\
&\geq -M\int_{\mathbb{R}^N} |\nabla u|^2 h_\delta \eta_k^2 \mathrm{d}x - \frac{M}{p}\int_{\mathbb{R}^N} \left(|\nabla u|^2+\delta^2\right)^{\frac{p}{2}}\eta_k^2 \mathrm{d}x \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^N} \left\langle b, \nabla\eta_k^2\right\rangle \left(|\nabla u|^2+\delta^2\right)^{\frac{p}{2}} \mathrm{d}x + \mathcal{O}_\delta^3 \\
&\geq -2M\int_{\mathbb{R}^N} \left(|\nabla u|^2+\delta^2\right)^{\frac{p}{2}}\eta_k^2 \mathrm{d}x + \mathcal{O}_\delta^2 + \mathcal{O}_\delta^3.
\end{aligned}
$$

Lastly, the third integral can be estimated as

$$
\begin{aligned}
I_3 &= -\int_{\mathbb{R}^N} cu\operatorname{div}(H_\delta(\nabla u))\eta_k^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \langle \nabla c, H_\delta(\nabla u)\rangle u\eta_k^2 + c|\nabla u|^2 h_\delta\eta_k^2 + cuh_\delta\left\langle \nabla\eta_k^2, \nabla u\right\rangle \mathrm{d}x \\
&\geq \int_{\mathbb{R}^N} \langle \nabla c, H_\delta(\nabla u)\rangle u\eta_k^2 \mathrm{d}x - M\int_{\mathbb{R}^N} \eta_k^2(|\nabla u|^2+\delta^2)^{\frac{p}{2}}\mathrm{d}x + \mathcal{O}_\delta^4 \\
&\geq -M\int_{\mathbb{R}^N} |u|\,|\nabla u|\, h_\delta\eta_k^2 \mathrm{d}x - M\int_{\mathbb{R}^N} \eta_k^2(|\nabla u|^2+\delta^2)^{\frac{p}{2}}\mathrm{d}x + \mathcal{O}_\delta^4
\end{aligned}
$$

where we have used that $c \in C_b^1(\mathbb{R}^N)$, inequality (2.2.6) and the Cauchy-Schwarz inequality. We continue by estimating the integrals $X_1, \ldots, X_4$. Let us start with the integral X_1. Let $\varepsilon \in (0,1)$, then, by the product rule, the Oleinik inequality, the Peter-Paul inequality with parameter ε and by inequality (2.2.6), we deduce

$$
X_1 = -\sum_{i,j,l=1}^N \int_{\mathbb{R}^N} \partial_{x_i}((\partial_{x_l} a_{ij})(\partial_{x_j} u))(\partial_{x_l} u)h_\delta\eta_k^2 \mathrm{d}x
$$

$$
\begin{aligned}
&= -\sum_{i,j,l=1}^{N}\int_{\mathbb{R}^N} (\partial_{x_i}\partial_{x_l}a_{ij})(\partial_{x_j}u)(\partial_{x_l}u)h_\delta\eta_k^2 + (\partial_{x_l}a_{ij})(\partial_{x_i}\partial_{x_j}u)(\partial_{x_l}u)h_\delta\eta_k^2\mathrm{d}x \\
&= -\sum_{i,j,l=1}^{N}\int_{\mathbb{R}^N} (\partial_{x_i}\partial_{x_l}a_{ij})(\partial_{x_j}u)(\partial_{x_l}u)h_\delta\eta_k^2\mathrm{d}x - \sum_{l=1}^{N}\int_{\mathbb{R}^N} \operatorname{tr}(\partial_{x_l}A\nabla^2 u)(\partial_{x_l}u)h_\delta\eta_k^2\mathrm{d}x \\
&\geq -M\int_{\mathbb{R}^N} |\nabla u|^2 h_\delta\eta_k^2\mathrm{d}x - \int_{\mathbb{R}^N} 2\varepsilon N^2 M\operatorname{tr}(\nabla^2 uA\nabla^2 U)h_\delta\eta_k^2 + \frac{1}{2\varepsilon}|\nabla u|^2\eta_k^2 h_\delta\mathrm{d}x \\
&\geq -\left(M+\frac{1}{2\varepsilon}\right)\int_{\mathbb{R}^N}\left(|\nabla u|^2+\delta^2\right)^{\frac{p}{2}}\eta_k^2\mathrm{d}x - 2\varepsilon N^2 M\int_{\mathbb{R}^N}\operatorname{tr}(\nabla^2 uA\nabla^2 u)h_\delta\eta_k^2\mathrm{d}x.
\end{aligned}
$$

We estimate the second integral X_2 using lemma B.0.2 as well as the Peter-Paul inequality with parameter ε

$$
\begin{aligned}
X_2 &= \sum_{i,j,l=1}^{N}\int_{\mathbb{R}^N} a_{ij}(\partial_{x_l}u)(\partial_{x_i}\eta_k^2)(\partial_{x_l}\partial_{x_j}u)h_\delta\mathrm{d}x \\
&= 2\sum_{i,j,l=1}^{N}\int_{\mathbb{R}^N} a_{ij}(\partial_{x_l}u)(\partial_{x_l}\partial_{x_j}u)(\partial_{x_i}\eta_k)h_\delta\eta_k\mathrm{d}x \\
&= 2\int_{\mathbb{R}^N}\left\langle\nabla u, A\nabla^2 u\nabla\eta_k\right\rangle\eta_k h_\delta\mathrm{d}x = 2\int_{\mathbb{R}^N}\left\langle A\nabla^2 u\nabla\eta_k,\nabla u\right\rangle\eta_k h_\delta\mathrm{d}x \\
&\geq -2\int_{\mathbb{R}^N}\sqrt{\langle A\nabla u,\nabla u\rangle|\nabla\eta_k|^2}\sqrt{\operatorname{tr}(\nabla^2 uA\nabla^2 u)\eta_k^2}h_\delta\mathrm{d}x \\
&\geq -\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\langle A\nabla u,\nabla u\rangle|\nabla\eta_k|^2 h_\delta\mathrm{d}x - \varepsilon\int_{\mathbb{R}^N}\operatorname{tr}(\nabla^2 uA\nabla^2 u)\eta_k^2 h_\delta\mathrm{d}x \\
&=: \mathcal{O}_\delta^6 - \varepsilon\int_{\mathbb{R}^N}\operatorname{tr}(\nabla^2 uA\nabla^2 u)\eta_k^2 h_\delta\mathrm{d}x.
\end{aligned}
$$

The third integral X_3 can be calculated as

$$
\begin{aligned}
X_3 &= \sum_{i,j,l=1}^{N}\int_{\mathbb{R}^N} a_{ij}(\partial_{x_l}\partial_{x_j}u)(\partial_{x_j}((\partial_{x_l}u)h_\delta))\eta_k^2\mathrm{d}x \\
&= (p-2)\sum_{i,j,l,n=1}^{N}\int_{\mathbb{R}^N} a_{ij}(\nabla^2 u)_{lj}\left(|\nabla u|^2+\delta^2\right)^{\frac{p-4}{2}}(\nabla^2 u)_{in}(\partial_{x_n}u)(\partial_{x_l}u)\eta_k^2\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N}\operatorname{tr}(\nabla^2 uA\nabla^2 u)h_\delta\eta_k^2\mathrm{d}x \\
&= (p-2)\int_{\mathbb{R}^N}\operatorname{tr}(\nabla^2 uA\nabla^2 u(\nabla u\otimes\nabla u))\left(|\nabla u|^2+\delta^2\right)^{\frac{p-4}{2}}\eta_k^2\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N}\operatorname{tr}(\nabla^2 uA\nabla^2 u)h_\delta\eta_k^2\mathrm{d}x.
\end{aligned}
$$

Moreover, using lemma B.0.2, the Peter-Paul inequality with parameter ε and the inequality

$$-|\nabla u|^2\left(|\nabla u|^2+\delta^2\right)^{\frac{p-4}{2}} \geq -h_\delta$$

we estimate the fourth integral X_4 as

$$\begin{aligned}
X_4 &= \sum_{i,j,l=1}^{N} \int_{\mathbb{R}^N} a_{ij}(\partial_{x_j}u)(\partial_{x_i}h_\delta)(\partial_{x_l}u)(\partial_{x_l}\eta_k^2)\mathrm{d}x \\
&= 2(p-2)\sum_{i,j,l,n=1}^{N} \int_{\mathbb{R}^N} a_{ij}(\partial_{x_j}u)(\nabla^2 u)_{in}(\partial_{x_n}u)(\partial_{x_l}u)\left(|\nabla u|^2+\delta^2\right)^{\frac{p-4}{2}}(\partial_{x_l}\eta_k)\eta_k\mathrm{d}x \\
&= 2(p-2)\int_{\mathbb{R}^N} \left\langle \nabla u, \nabla^2 u A\nabla u\right\rangle \left\langle \nabla\eta_k, \nabla u\right\rangle \left(|\nabla u|^2+\delta^2\right)^{\frac{p-4}{2}}\eta_k\mathrm{d}x \\
&\geq -2\int_{\mathbb{R}^N}\sqrt{\langle A\nabla u,\nabla u\rangle (p-2)^2|\nabla u|^2|\nabla\eta_k|^2}\sqrt{\operatorname{tr}(\nabla^2 uA\nabla^2 u)|\nabla u|^2}\left(|\nabla u|^2+\delta^2\right)^{\frac{p-4}{2}}\eta_k^2\mathrm{d}x \\
&\geq -\frac{(p-2)^2}{\varepsilon}\int_{\mathbb{R}^N}\langle A\nabla u,\nabla u\rangle h_\delta|\nabla\eta_k|^2\mathrm{d}x - \varepsilon\int_{\mathbb{R}^N}\operatorname{tr}(\nabla^2 uA\nabla^2 u)\eta_k^2 h_\delta\mathrm{d}x \\
&=: \mathcal{O}_\delta^5 - \varepsilon\int_{\mathbb{R}^N}\operatorname{tr}(\nabla^2 uA\nabla^2 u)\eta_k^2 h_\delta\mathrm{d}x.
\end{aligned}$$

Thus, all estimates on $X_1,\dots,X_4$ together give

$$\begin{aligned}
I_1 \geq{}& \sum_{i=1}^{6}\mathcal{O}_\delta^i - \left(M+\frac{1}{2\varepsilon}\right)\int_{\mathbb{R}^N}\left(|\nabla u|^2+\delta^2\right)^{\frac{p}{2}}\eta_k^2\mathrm{d}x \\
&+\left(1-2\varepsilon N^2 M-2\varepsilon\right)\int_{\mathbb{R}^N}\operatorname{tr}(\nabla^2 uA\nabla^2 u)h_\delta\eta_k^2\mathrm{d}x \\
&+(p-2)\int_{\mathbb{R}^N}\operatorname{tr}\left(\nabla^2 uA\nabla^2 u(\nabla u\otimes\nabla u)\right)\eta_k^2\left(|\nabla u|^2+\delta^2\right)^{\frac{p-4}{2}}\mathrm{d}x.
\end{aligned}$$

Choosing $\varepsilon>0$ small enough, we may assume $(1-2\varepsilon N^2M-2\varepsilon)>0$. It is $(\nabla u\otimes\nabla u)_{ij}\leq |\nabla u|^2$, which shows that the maximal eigenvalue of $\nabla u\otimes\nabla u$ is bounded by $|\nabla^2 u|$. By the positive semidefiniteness of A, it holds $\nabla^2 uA\nabla^2 u\geq 0$ in $\mathbb{R}^N$ and therefore, by [WKH86, Lemma 1], we deduce

$$\operatorname{tr}(\nabla^2 uA\nabla^2 u(\nabla u\otimes\nabla u))\leq|\nabla u|^2\operatorname{tr}(\nabla^2 uA\nabla^2 u).$$

Furthermore, since $\nabla u\otimes\nabla u$ is symmetric positive semidefinite it holds

$$0\leq\operatorname{tr}(\nabla^2 uA\nabla^2 u(\nabla u\otimes\nabla u))\leq|\nabla u|^2\operatorname{tr}(\nabla^2 uA\nabla^2 u). \tag{2.2.7}$$

Consequently,

$$\begin{aligned}\left\langle Eu, \eta_k^2(-\operatorname{div}(H_\delta(\nabla u))\right\rangle \geq \sum_{i=1}^{6} \mathcal{O}_\delta^i - \left(4M + \frac{1}{2\varepsilon}\right) \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \delta^2\right)^{\frac{p}{2}} \eta_k^2 \mathrm{d}x \\ - M \int_{\mathbb{R}^N} |u| \, |\nabla u| \, h_\delta \eta_k^2 \mathrm{d}x \\ + \left(p - 1 - 2\varepsilon N^2 M - 2\varepsilon\right) \int_{\mathbb{R}^N} \operatorname{tr}(\nabla^2 u A \nabla^2 u (\nabla u \otimes \nabla u)) \left(|\nabla u|^2 + \delta^2\right)^{\frac{p-4}{2}} \eta_k^2 \mathrm{d}x.\end{aligned}$$

If necessary, we reduce $\varepsilon > 0$ such that the factor of the last integral remains positive. By equation (2.2.7), we may estimate the respective integral by zero from below. Due to the assumption that $\nabla(Eu) \in L^p(\mathbb{R}^N)$ we may integrate by parts to deduce

$$\begin{aligned}\left\langle \eta_k^2 \nabla(Eu) + Eu \nabla \eta_k^2, H_\delta \right\rangle \geq \sum_{i=1}^{6} \mathcal{O}_\delta^i - \left(4M + \frac{1}{2\varepsilon}\right) \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \delta^2\right)^{\frac{p}{2}} \eta_k^2 \mathrm{d}x \\ - M \int_{\mathbb{R}^N} |u| \, |\nabla u| \, h_\delta \eta_k^2 \mathrm{d}x.\end{aligned}$$

To conclude, we want to to take the limits $k \to \infty$ and $\delta \to 0$. We recall that we need to consider the limit $\delta \to 0$ only if $p < 2$. By theorem of dominated convergence and the theorem of monotone convergence, we conclude

$$\begin{aligned}\left\langle \eta_k^2 \nabla(Eu) + Eu \nabla \eta_k^2, |\nabla u|^{p-2} \nabla u \right\rangle \geq \sum_{i=1}^{6} \mathcal{O}^i - \left(4M + \frac{1}{2\varepsilon}\right) \int_{\mathbb{R}^N} |\nabla u|^p \eta_k^2 \mathrm{d}x \\ - M \int_{\mathbb{R}^N} |u| \, |\nabla u|^{p-1} \eta_k^2 \mathrm{d}x\end{aligned}$$

as $\delta \to 0$. Using the Hölder inequality in the last term of the right-hand side shows

$$\begin{aligned}\left\langle \eta_k^2 \nabla(Eu) + Eu \nabla \eta_k^2, |\nabla u|^{p-2} \nabla u \right\rangle &\geq \sum_{i=1}^{6} \mathcal{O}^i - \left(4M + \frac{1}{2\varepsilon}\right) \int_{\mathbb{R}^N} |\nabla u|^p \eta_k^2 \mathrm{d}x \\ &\quad - M \int_{\mathbb{R}^N} |u| \, |\nabla u|^{p-1} \eta_k^2 \mathrm{d}x \\ &\geq \sum_{i=1}^{6} \mathcal{O}^i - \left(4M + \frac{1}{2\varepsilon}\right) \int_{\mathbb{R}^N} |\nabla u|^p \eta_k^2 \mathrm{d}x \\ &\quad - M \, \|u\|_{p,\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |\nabla u|^p \eta_k^{\frac{2p}{p-1}} \mathrm{d}x\right)^{\frac{p-1}{p}}\end{aligned}$$

for all $k \in \mathbb{N}$. Let us now consider the limit $k \to \infty$. We note that the terms $\mathcal{E}^1, \dots, \mathcal{E}^6$

converge to 0 as $k \to \infty$. This is due to the finiteness of A^∞, b^∞ and c^∞, the boundedness of ∇c, the fact that $u \in W^{1,p}(\mathbb{R}^N)$ and due to the behavior of $\nabla \eta_k$. An application of the theorem of dominated convergence shows the claimed convergence. Analog arguments have already been used in the proof of proposition 2.2.1.

Since $\nabla u \in L^p(\mathbb{R}^N)$, it holds $|\nabla u|^{p-2} \nabla u \in L^q(\mathbb{R}^N)$, where q is the dual exponent of p. Due to $Eu \in L^p(\mathbb{R}^N)$, we conclude that $Eu\nabla\eta_k^2 \to 0$ in $L^p(\mathbb{R}^N)$. This shows that $\left\langle Eu\nabla\eta_k^2, |\nabla u|^{p-2} \nabla u\right\rangle \to 0$ by duality of $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$. We deduce

$$\left\langle \nabla(Eu), |\nabla u|^{p-2} \nabla u\right\rangle \geq -\left(4M + \frac{1}{2\varepsilon}\right) \|\nabla u\|_{p,\mathbb{R}^N}^p - M \|u\|_{p,\mathbb{R}^N} \|\nabla u\|_{p,\mathbb{R}^N}^{p-1}.$$

Furthermore, by Young's inequality, we estimate

$$\begin{aligned}\left\langle \nabla(Eu), |\nabla u|^{p-2} \nabla u\right\rangle &\geq -\left(4M + \frac{1}{2\varepsilon}\right) \|\nabla u\|_{p,\mathbb{R}^N}^p - \frac{M}{p} \|u\|_{p,\mathbb{R}^N}^p - \frac{M(p-1)}{p} \|\nabla u\|_{p,\mathbb{R}^N}^p \\ &\geq -\left(5M + \frac{1}{2\varepsilon}\right) \|\nabla u\|_{p,\mathbb{R}^N}^p - M \|u\|_{p,\mathbb{R}^N}^p.\end{aligned}$$

Since $\pm u \in C^2(\mathbb{R}^N) \cap D(\mathcal{E}_p)$, by proposition 2.2.1 applied to $\pm u$, we also deduce that

$$\left\langle Eu, |u|^{p-2} u\right\rangle \geq -2M \left\langle u, |u|^{p-2} u\right\rangle.$$

By adding the last two inequalities, we finally conclude

$$\begin{aligned}\left\langle Eu, |u|^{p-2} u\right\rangle + \left\langle \nabla(Eu), |\nabla u|^{p-2} \nabla u\right\rangle &\geq -\left(5M + \frac{1}{2\varepsilon}\right) \|\nabla u\|_{p,\mathbb{R}^N}^p - 3M \left\langle u, |u|^{p-2} u\right\rangle \\ &\geq -\left(5M + \frac{1}{2\varepsilon}\right) \|u\|_{1,p,\mathbb{R}^N}^p := -\lambda_2 \|u\|_{1,p,\mathbb{R}^N}^p. \quad \square\end{aligned}$$

2.2.6 Quasi-m-dispersiveness of an intermediate operator

For technical reasons we introduce the intermediate operator E_p^i with domain

$$D(E_p^i) = \{u \in W^{1,p}(\mathbb{R}^N) \cap D(\mathcal{E}_p) \mid \langle A\nabla u, \nabla u\rangle^{\frac{1}{2}} \in L^p(\mathbb{R}^N)\}$$

and $E_p^i u = \mathcal{E}_p u$ for all $u \in D(E_p^i)$. In particular, it holds $C_c^\infty(\mathbb{R}^N) \subset D(E_p^i) \subset D(\mathcal{E}_p)$.

Proposition 2.2.13. As in proposition 2.2.12, we assume that $c \in C^1(\mathbb{R}^N)$ with bounded first derivatives. Under this additional assumption there exists a constant $\lambda_3 > 0$ such that $\mathcal{R}(\lambda + E_p^i)$ is dense in $L^p(\mathbb{R}^N)$ for all $\lambda > \lambda_3 + 1$.

Proof. We choose $\tilde{\lambda}_3$ larger than the maximum of the constants λ_0, λ_1 and λ_2 given in the previous statements for the operators $E_{\varepsilon,0}$ and $E_{\varepsilon,1}$ and set $\lambda_3 = \tilde{\lambda}_3 + 1 - p^{-1}$. The constant $\tilde{\lambda}_3$ can be chosen independently of $\varepsilon \in (0,1)$ as seen in inequality (2.2.4).

Let $g \in C_c^\infty(\mathbb{R}^N)$. We want to calculate a solution $u \in D(E_p^i)$ of the equation $(\lambda + E_p^i)u = g$. Due to the choice of $\lambda > \lambda_3 + 1$, we know that there exists a weak solution to the corresponding regularized problem. In particular, by elliptic regularity, there are functions $u_{\varepsilon,0,p}, u_{\varepsilon,1,p} \in C^\infty \cap D(\mathcal{E}_{\varepsilon,p})$ which are solutions of

$$(\lambda + E_{\varepsilon,0})u_{\varepsilon,0,p} = g \tag{2.2.8}$$

$$(\lambda + E_{\varepsilon,1})u_{\varepsilon,1,p} = (1+|x|)^{\frac{1}{2}} g =: g_1. \tag{2.2.9}$$

By lemma 2.2.10, it holds $u_{\varepsilon,0,p}, u_{\varepsilon,0,p} \in W^{1,1} \cap W^{1,\infty}(\mathbb{R}^N) \subset W^{1,p}(\mathbb{R}^N)$. Moreover, arguing as in the proof of lemma 2.2.10, it is $u_{\varepsilon,1,p} = (1+|x|)^{\frac{1}{2}} u_{\varepsilon,0,p}$ and

$$\nabla u_{\varepsilon,1,p} = \nabla \left((1+|x|)^{\frac{1}{2}} u_{\varepsilon,0,p} \right) \in L^p(\mathbb{R}^N).$$

Differentiating equation (2.2.9), we deduce $\nabla(E_{\varepsilon,1} u_{\varepsilon,1,p}) = \nabla g_1 - \lambda \nabla u_{\varepsilon,1,p} \in L^p(\mathbb{R}^N)$. Since the zeroth order term of $E_{\varepsilon,1}$ is in $C_b^1(\mathbb{R}^N)$, we may apply proposition 2.2.12 to deduce

$$\langle g_1 - \lambda u_{\varepsilon,1,p}, |u_{\varepsilon,1,p}|^{p-2} u_{\varepsilon,1,p} \rangle + \langle \nabla g_1 - \lambda \nabla u_{\varepsilon,1,p}, |\nabla u_{\varepsilon,1,p}|^{p-2} \nabla u_{\varepsilon,1,p} \rangle \geq -\tilde{\lambda}_3 \left\| u_{\varepsilon,1,p} \right\|_{1,p,\mathbb{R}^N}^p ,$$

whence

$$\begin{aligned}
(\lambda - \tilde{\lambda}_3) \left\| u_{\varepsilon,1,p} \right\|_{1,p,\mathbb{R}^N}^p &\leq \langle g_1, |u_{\varepsilon,1,p}|^{p-2} u_{\varepsilon,1,p} \rangle + \langle \nabla g_1, |\nabla u_{\varepsilon,1,p}|^{p-2} \nabla u_{\varepsilon,1,p} \rangle \\
&\leq \left\| g_1 \right\|_{p,\mathbb{R}^N} \left\| u_{\varepsilon,1,p} \right\|_{p,\mathbb{R}^N}^{p-1} + \left\| \nabla g_1 \right\|_{p,\mathbb{R}^N} \left\| \nabla u_{\varepsilon,1,p} \right\|_{p,\mathbb{R}^N}^{p-1} \\
&\leq \frac{1}{p} \left\| g_1 \right\|_{1,p,\mathbb{R}^N}^p + \frac{p-1}{p} \left\| u_{\varepsilon,1,p} \right\|_{p,\mathbb{R}^N}^p \\
&\leq \left\| g_1 \right\|_{1,p,\mathbb{R}^N}^p + \frac{p-1}{p} \left\| u_{\varepsilon,1,p} \right\|_{p,\mathbb{R}^N}^p
\end{aligned}$$

by Young's inequality. This shows

$$\left\| u_{\varepsilon,1,p} \right\|_{1,p,\mathbb{R}^N} \leq \left(\frac{1}{\lambda - (\tilde{\lambda}_3 + 1 - p^{-1})} \right)^{\frac{1}{p}} \left\| g_1 \right\|_{1,p,\mathbb{R}^N} \leq \frac{1}{\lambda - \lambda_3} \left\| g_1 \right\|_{1,p,\mathbb{R}^N},$$

since $\lambda - \lambda_3 > 1$. In particular, the sequence $(u_{\varepsilon,1,p})_{\varepsilon \in (0,1)}$ is uniformly bounded in $W^{1,p}(\mathbb{R}^N)$. Similarly, it holds

$$\left\| u_{\varepsilon,0,p} \right\|_{1,p,\mathbb{R}^N} \leq \frac{1}{\lambda - \lambda_3} \left\| g \right\|_{1,p,\mathbb{R}^N}. \tag{2.2.10}$$

Hence, the sequence $(u_{\varepsilon,0,p})_{\varepsilon\in(0,1)}$ is uniformly bounded in $W^{1,p}(\mathbb{R}^N)$, too.

Consequently the sequence $(E_{\varepsilon,0,p}u_{\varepsilon,0,p})_{\varepsilon\in(0,1)}$ is uniformly bounded in $W^{1,p}(\mathbb{R}^N)$. Due to the reflexivity of $W^{1,p}(\mathbb{R}^N)$ there exist weakly convergent subsequences

$$u_{\varepsilon,0,p} \rightharpoonup u, \quad \sqrt{1+|x|^2}u_{\varepsilon,0,p} = u_{\varepsilon,1,p} \rightharpoonup v, \quad \mathcal{E}_{\varepsilon,0,p}u_{\varepsilon,0,p} \rightharpoonup w$$

with corresponding limit functions $u, v, w \in W^{1,p}(\mathbb{R}^N)$. By equation (2.2.8), it must hold $w = g - \lambda u$ and $v = \sqrt{1+|x|^2}u$. To conclude that u is indeed a solution of $(\lambda + E_p^i)u = g$, it remains to show that $u \in D(E_p^i)$ and that $E_p^i u = w$. It is $u \in W^{1,p}(\mathbb{R}^N)$ and in particular $\nabla u \in L^p(\mathbb{R}^N)$. Since $\sqrt{1+|x|^2}u = v \in W^{1,p}(\mathbb{R}^N)$, it follows from

$$\sqrt{1+|x|^2}\nabla u = \nabla v - \frac{xu}{\sqrt{1+|x|^2}} \in L^p(\mathbb{R}^N)$$

and from the fact that a_{ij} grows at most of quadratic order, that $\sqrt{\langle A\nabla u, \nabla u\rangle} \in L^p(\mathbb{R}^N)$.

Next, we are going to show that $\langle u_{\varepsilon,0,p}, E_{\varepsilon,0}^T\varphi\rangle \to \langle u, E^T\varphi\rangle$ as $\varepsilon \to 0$ for every test function $\varphi \in C_c^\infty(\mathbb{R}^N)$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$. The sequence $(E_{\varepsilon,0}^T\varphi)_{\varepsilon>0}$ is uniformly bounded in the dual space $L^q(\mathbb{R}^N)$ and it holds $E_{\varepsilon,0}^T\varphi \to E^T\varphi$ pointwise, since $\varphi \in C_c^\infty(\mathbb{R}^N)$. Due to the boundedness in $L^q(\mathbb{R}^N)$ and by the theorem of dominated convergence, this convergence holds in $L^q(\mathbb{R}^N)$ as well. Weak convergence in $W^{1,p}(\mathbb{R}^N)$ implies the weak convergence in $L^p(\mathbb{R}^N)$. In particular, $u_{\varepsilon,0,p} \rightharpoonup u$ in $L^p(\mathbb{R}^N)$. We conclude that $\langle u_{\varepsilon,0,p}, E_{\varepsilon,0}^T\varphi\rangle \to \langle u, E^T\varphi\rangle$ as $\varepsilon \to 0$ by duality of $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$. From $E_{\varepsilon,0}u_{\varepsilon,0,p} \rightharpoonup w$ in $L^p(\mathbb{R}^N)$ we deduce

$$\langle u, E^T\varphi\rangle = \lim_{\varepsilon\to 0}\langle u_{\varepsilon,0,p}, E_{\varepsilon,0}^T\varphi\rangle = \lim_{\varepsilon\to 0}\langle E_{\varepsilon,0}u_{\varepsilon,0,p}, \varphi\rangle = \langle w, \varphi\rangle$$

and thus $w = E_p^i u \in L^p(\mathbb{R}^N)$ and $(\lambda + E_p^i)u = g$. We conclude that $u \in D(E_p^i)$, whence $C_c^\infty(\mathbb{R}^N) \subset \mathcal{R}(\lambda + E_p^i)$. This shows that the range of $\lambda + E_p^i$ is dense in $L^p(\mathbb{R}^N)$. □

Proposition 2.2.14. It holds $(E_p^i, D(E_p^i)) \subset \overline{(E, C_c^\infty(\mathbb{R}^N))}^p$.

Proof. To prove the claim, we are going to show that every $u \in D(E_p^i)$ can be approximated by a sequence of functions $(u_k) \subset C_c^\infty(\mathbb{R}^N)$ such that $u_k \to u$ and $Eu_k \to E_i^p u$ in $L^p(\mathbb{R}^N)$. This shows that $u \in D(\overline{E}^p)$.

As a step towards this claim, we first show that every function in $D(E_i^p)$ can be approximated by functions $D(E_p^i) \cap L_c^p(\mathbb{R}^N)$ in above sense, where

$$L_c^p(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) \mid \operatorname{supp} u \subset \mathbb{R}^n \text{ is bounded}\}.$$

Let $u \in D(E_p^i)$ and define the sequence $u_k = \eta_k u$ where η_k denotes the sequence of cutoff functions from section A.3. Since $\eta_k \in C_c^\infty(\mathbb{R}^N)$ and $0 \leq \eta_k \leq 1$, it holds $u_k \in D(E_p^i) \cap L_c^p(\mathbb{R}^N)$ with

$$E_p^i u_k = \eta_k E_p^i u - 2\langle A\nabla\eta_k, \nabla u\rangle - \sum_{i,j=1}^N \partial_{x_i} a_{ij} \partial_{x_j} \eta_k u - \operatorname{tr}(A\nabla^2\eta_k)u + \langle b, \nabla\eta_k\rangle u.$$

This can be shown by performing partial integration in the integral $\langle u\eta_k, E^T\varphi\rangle$, while using the fact that $u \in W^{1,p}(\mathbb{R}^N)$.

It remains to show that $E_p^i u_k \to E_p^i u$ in $L^p(\mathbb{R}^N)$. It holds $\eta_k E_p^i u \to E_p^i u$ in $L^p(\mathbb{R}^N)$, since $E_p^i u \in L^p(\mathbb{R}^N)$. We want to prove that all the other terms converge to zero. Due to the boundedness assumption made on the derivatives of the coefficients, we see that $\partial_{x_i} a_{ij}$ and b_j grow at most linearly in $|x|$ and a_{ij} grows at most of quadratic order in $|x|$. As presented in section A.3, the derivative of the cutoff function $\nabla\eta_k$ decays of order than k^{-1} and its support is contained in $B_{3k}(0)$. The second derivatives $\nabla^2\eta_k$ decay faster than k^{-2} while their support is contained in $B_{3k}(0)$, too. We conclude

$$\left|\sum_{i,j=1}^N \partial_{x_i} a_{ij} \partial_{x_i} \eta_k u\right| \leq C\,|u|$$

for a constant $C \geq 0$. Given $x \in \mathbb{R}^N$, it holds $\eta_k(x) \to 0$ and $\nabla\eta_k(x) \to 0$ as $k \to \infty$. Therefore, $\partial_{x_i} a_{ij} \partial_{x_i} \eta_k u \to 0$ pointwise boundedly. Using that $u \in L^p(\mathbb{R}^N)$, we deduce that $\partial_{x_i} a_{ij} \partial_{x_i} \eta_k u \to 0$ in $L^p(\mathbb{R}^N)$ as an application of the theorem of dominated convergence. The convergence of the fourth and fifth terms to zero follow by similar arguments. For the second term we note that

$$|\langle A\nabla\eta_k, \nabla u\rangle| \leq \sqrt{\langle A\nabla\eta_k, \nabla\eta_k\rangle}\sqrt{\langle A\nabla u, \nabla u\rangle} \in L^p(\mathbb{R}^N),$$

since $u \in D(E_p^i)$ and $\langle A\nabla\eta_k, \nabla\eta_k\rangle \in L^\infty(\mathbb{R}^N)$. Arguing as before, it holds $\langle A\nabla\eta_k, \eta u\rangle \to 0$ in $L^p(\mathbb{R}^N)$ by the theorem of dominated convergence and therefore we deduce $E_p^i u_k \to E_p^i u$ in $L^p(\mathbb{R}^N)$.

Next, we choose any $u \in D(E_p^i) \cap L_c(\mathbb{R}^N)$ and define $u_k = (\omega_k * u) \in C_c^\infty(\mathbb{R}^N) \subset L_c^p(\mathbb{R}^N) \cap D(E_p^i)$. By theorem A.3.5, it holds $u_k \to u$ in $L^p(\mathbb{R}^N)$ as well as $Eu_k \to E_p^i u$ in $L^p(\mathbb{R}^N)$.

Finally, let $u \in D(E_p^i)$. We choose a sequence $(u_k)_{k\in\mathbb{N}} \subset D(E_p^i) \cap L_c(\mathbb{R}^N)$ such that $u_k \to u$ and $E_p^i u_k \to Eu$ as $k \to \infty$. Moreover, for every $k \in \mathbb{N}$ we choose a sequence $(u_{k,l})_{l\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^N)$ such that $u_{k,l} \to u_k$ and $Eu_{k,l} \to E_p^i u_k$ as $l \to \infty$. To conclude, we choose the sequence $(u_{k,k})_{k\in\mathbb{N}}$, which satisfies $u_{k,k} \to u$ and $Eu_{k,k} \to E_p^i u$. As explained, this implies $u \in D(\overline{E}^p)$ and henceforth that $(E_p^i, D(E_p^i)) \subset \overline{(E, C_c^\infty(\mathbb{R}^N))}^p$. □

Corollary 2.2.15. We assume that $c \in C_b^1(\mathbb{R}^N)$ with bounded derivatives. Let $\lambda > \lambda_3 + 1$, where λ_3 is given by proposition 2.2.13. It holds $\mathcal{R}(\lambda + \overline{E}^p) = L^p(\mathbb{R}^N)$.

Proof. By proposition 2.2.13, we deduce that $\mathcal{R}(\lambda + E_p^i)$ is dense in $L^p(\mathbb{R}^N)$. The claim follows using proposition 2.2.14 and noting that

$$L^p(\mathbb{R}^N) = \overline{\mathcal{R}(\lambda + E_p^i)} \subset \overline{\mathcal{R}(\lambda + \overline{E}^p)} = \mathcal{R}(\lambda + \overline{E}^p) \subset L^p(\mathbb{R}^N),$$

since as seen in the proof of proposition 2.2.3 the range of $\lambda + \overline{E}^p$ is closed. □

2.2.7 Quasi-m-dispersiveness of the negative maximal realization

Theorem 2.2.16. *Let $p \in (1, \infty)$ and denote by $(E, C_c^\infty(\mathbb{R}^N))$ and $(\mathcal{E}_p, D(\mathcal{E}_p))$ the minimal and the maximal realization of a degenerate second order elliptic differential operator satisfying the assumptions (A1) and (A2). Then $(E, C_c^\infty(\mathbb{R}^N))$ is closable in $L^p(\mathbb{R}^N)$ and its closure is given by $(\mathcal{E}_p, D(\mathcal{E}_p))$. The negative maximal realization $(-\mathcal{E}_p, D(\mathcal{E}_p))$ is quasi-m-dispersive.*

Proof. To apply the previous results, we introduce the perturbed operator $\underline{E}$ given by $\underline{E} = -\operatorname{div}(A\nabla u) + \langle b, \nabla u\rangle$. It satisfies the assumptions of corollary 2.2.15, since $c = 0$. We choose $\lambda > \max\{3M, \lambda_3 + 1\}$, where $M = \max\{A^\infty, b^\infty, c^\infty\}$ and $\lambda_3 > 0$ is the constant given by proposition 2.2.13 corresponding to the operator $\underline{E}$. By definition A.1.5, it follows that $-\overline{\underline{E}}^p$ is quasi-m-dispersive. Furthermore, by proposition 2.2.1, we may estimate the operator norm of the resolvent as $\left\|(\lambda + \overline{\underline{E}}^p)^{-1}\right\| \le \lambda - 2M$. Here we have used the equivalent characterization of accretiveness given by proposition A.1.6 together with proposition A.1.10.

We introduce the multiplication operator $Uu = cu$ which is a bounded linear operator on $L^p(\mathbb{R}^N)$. It holds $\|c\|_{\infty,\mathbb{R}^N} \le M$ and therefore $\|U\| \le M$. Moreover, for every $\varphi \in C_c^\infty(\mathbb{R}^N)$ it holds

$$\lambda\varphi + E\varphi = \lambda\varphi + \underline{E}\varphi + U\varphi = (\operatorname{Id} + U(\lambda + \underline{E})^{-1})(\lambda + \underline{E})\varphi.$$

From the inequality

$$\left\|U(\lambda + \underline{E})^{-1}\right\| \le \frac{M}{\lambda - 2M} < 1$$

and the Neumann series we are able to deduce that the operator $(\operatorname{Id} + U(\lambda + \underline{E})^{-1})$ is invertible in $L^p(\mathbb{R}^N)$. This shows that $\mathcal{R}(\lambda + \overline{E}^p) = \mathcal{R}(\lambda + \overline{\underline{E}}^p)$, since the operators E and $\underline{E}$ are closable in $L^p(\mathbb{R}^N)$. Using corollary 2.2.15, we deduce

$$L^p(\mathbb{R}^N) = \mathcal{R}(\lambda + \overline{\underline{E}}^p) = \mathcal{R}(\lambda + \overline{E}^p) \subset L^p(\mathbb{R}^N)$$

and thus the closure of $(-E, C_c^\infty(\mathbb{R}^N))$ in $L^p(\mathbb{R}^N)$ is quasi-m-dispersive. Finally, we conclude $L^p(\mathbb{R}^N) = \mathcal{R}(\lambda + \overline{E}^p) = \mathcal{R}(\lambda + \mathcal{E}_p)$. It holds $E \subset \mathcal{E}_p$ and thus it must be $\overline{(E, C_c^\infty(\mathbb{R}^N))}^p = (D(\mathcal{E}_p), \mathcal{E}_p)$ as a consequence of the quasi-dispersiveness, as seen at the end of the proof of corollary 2.2.7. □

Corollary 2.2.17. The maximal realization $(\mathcal{E}_p, D(\mathcal{E}_p))$ is quasi-m-accretive and the minimal realization $(E, C_c^\infty(\mathbb{R}^N))$ is essentially quasi-m-accretive.

Proof. This is the consequence of lemma 2.2.6 and the latter theorem. □

2.2.8 The semigroup generated by the maximal realization

For the readers convenience we recall the assumptions and the definition of the differential operators $E, \mathcal{E}_p$. We are dealing with differential operators of the form

$$Eu = -\operatorname{div}(A\nabla u) + \langle b, \nabla u\rangle + c$$

with coefficients A, b and c satisfying the following assumptions

(A1) $A \in C^2(\mathbb{R}^N; \mathbb{R}^{N\times N})$ with bounded second derivatives, $b \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ with bounded derivatives as well as $c \in L^\infty(\mathbb{R}^N)$.

(A2) For all $x \in \mathbb{R}^N$ the matrix $A(x)$ is assumed to be positive semidefinite.

Let $p \in (1, \infty)$. The minimal realization is the classical differential operator $(E, C_c^\infty(\mathbb{R}^N))$ and the maximal realization $(\mathcal{E}_p, D(\mathcal{E}_p))$ is to be understood in the distributional sense on its domain

$$D(\mathcal{E}_p) = \{u \in L^p(R^N) \mid Eu \in L^p(\mathbb{R}^N)\}.$$

Theorem 2.2.18. *The negative maximal realization $(-\mathcal{E}_p, D(\mathcal{E}_p))$ is the generator of a strongly continuous, quasi-contractive and positive semigroup.*

Proof. At this point, this is merely a consequence of theorem A.1.9 and theorem 2.2.16. □

Corollary 2.2.19. If $\lambda \in \rho(-\mathcal{E}_p)$, then for every $g \in L^p(\mathbb{R}^N)$ there exists a unique distributional solution u of the equation

$$(\lambda + \mathcal{E}_p)u = g.$$

Moreover, if $g \geq 0$, then $u \geq 0$.

Proof. This is a direct consequence of the generator property of $(-\mathcal{E}_p, D(\mathcal{E}_p))$. Note that positivity of the semigroup implies positivity of the resolvent by proposition A.1.8. □

Theorem 2.2.20. *For every initial datum $f \in L^p(\mathbb{R}^N)$ the Cauchy problem*

$$\begin{cases} \partial_t u = -Eu, & t > 0 \\ u(0) = f \end{cases}$$

admits a unique weak solution $u \in C([0,\infty); L^p(\mathbb{R}^N))$. The function u is a strong solution if and only if $f \in D(\mathcal{E}_p)$. The corresponding semigroup $(T(t))_{t\geq 0}$ is positive.

Proof. This is the direct consequence of theorem 2.2.18, proposition A.1.16 and proposition A.1.17. □

Theorem 2.2.21. *If $\lambda > \max\{3M, \lambda_3 + 1\}$, then $W^{1,p}(\mathbb{R}^N)$ is an invariant subspace of the operator $(\lambda + \mathcal{E}_p)^{-1}$ and of the semigroup $(T(t))_{t\geq 0}$. Furthermore, it holds*

$$\left\|(\lambda + \mathcal{E}_p)^{-1} u\right\|_{1,p,\mathbb{R}^N} \leq \frac{1}{\lambda - \lambda_3} \|u\|_{1,p,\mathbb{R}^N} \text{ and } \|T(t)f\|_{1,p,\mathbb{R}^N} \leq \exp(\lambda_3 t) \|f\|_{1,p,\mathbb{R}^N}$$

for all $u, f \in W^{1,p}(\mathbb{R}^N)$.

Proof. If $g \in C_c^\infty(\mathbb{R}^N)$, we have seen that a solution of the equation $\lambda u + Eu = g$ can be constructed as the weak $W^{1,p}(\mathbb{R}^N)$ limit of functions $u_{\varepsilon,0,p} \in W^{1,p}(\mathbb{R}^N)$. Moreover, by inequality (2.2.10), we know that these function are uniformly bounded. By the lower semicontinuity of the norm with respect to weak convergence, we deduce

$$\left\|(\lambda + E)^{-1} g\right\|_{1,p,\mathbb{R}^N} = \|u\|_{1,p,\mathbb{R}^N} \leq \liminf_{\varepsilon \to 0} \|u_{\varepsilon,0,p}\|_{1,p,\mathbb{R}^N} \leq \frac{1}{\lambda - \lambda_3} \|g\|_{1,p,\mathbb{R}^N} .$$

Thus, by density of $C_c^\infty(\mathbb{R}^N)$ in $W^{1,p}(\mathbb{R}^N)$, we conclude the estimate for the resolvent. The estimate for the semigroup is the consequence of the Yosida approximation for the semigroup. Let $f \in W^{1,p}(\mathbb{R}^N)$, then

$$T(t)f = \lim_{n\to\infty} \left(1 + \frac{t}{n}\mathcal{E}_p\right)^{-n} f$$

for all $t > 0$. Consequently,

$$\|T(t)f\|_{1,p,\mathbb{R}^N} \leq \limsup_{n\to\infty} \left(\frac{t}{n}\right)^{-n} \frac{1}{\left(\frac{n}{t} - \lambda_3\right)^n} \|f\|_{1,p,\mathbb{R}^N} = \exp(\lambda_3 t) \|f\|_{1,p,\mathbb{R}^N} .$$ □

Remark 2.2.22. Let us compare the results presented in this section with the results obtained in the article [Iga74]. Therein the well-posedness under similar assumptions is proven in the case $p = 2$. At first glance this approach seems much shorter and more elegant. However, it is not as powerful as the approach presented in this chapter. In contrast to [Iga74] we derive the positiveness of the semigroup $T(t)$. Furthermore, we consider every $p \in (1, \infty)$ and prove that $W^{1,p}(\mathbb{R}^N)$ is an invariant subspace to the semigroup.

2.3 The Cauchy problem on bounded domains

In the previous section we have seen how to treat degenerate elliptic-parabolic second order partial differential equations in $\mathbb{R}^N$. The aim of this section is to present an outlook on the theory of degenerate elliptic-parabolic second order partial differential equations on bounded domains $\Omega \subset \mathbb{R}^N$. The section is based on the book [Ole73] and on [WYW06, Chapter 13].

2.3.1 Generalized Dirichlet boundary conditions

We start with the presentation of a suitable framework for the theory of degenerate second order elliptic partial differential equations on bounded domains. Let $\Omega \subset \mathbb{R}^N$ be a bounded, connected and open set with piecewise C^1-boundary and outer unit normal $n(x)$. We are interested in the second order partial differential operator given by

$$Eu = -\operatorname{tr}(A\nabla^2 u) + \langle b, \nabla u\rangle + cu \tag{2.3.1}$$

in trace form or written in divergence form as

$$Eu = -\operatorname{div}(A\nabla u) + \langle b, \nabla u\rangle + \sum_{i,j=1}^{N} \partial_{x_j} a_{ij} \partial_{x_i} u + cu.$$

For the sake of simplicity we assume that the coefficient functions are smooth up to the boundary, i.e. $A \in C^\infty(\overline{\Omega})$, $b \in C^\infty(\overline{\Omega})$ and $c \in C^\infty(\overline{\Omega})$. Moreover, we assume the positive semidefiniteness of the matrix A, i.e. $\langle A(x)\xi, \xi\rangle \geq 0$ for all $\xi \in \mathbb{R}^N$ and any $x \in \overline{\Omega}$.

The boundary $\Sigma = \partial\Omega$ of Ω will be separated into different parts. We denote by $\Sigma^0 = \{x \in \partial\Omega \mid \langle An(x), n(x)\rangle = 0\}$. Furthermore, we introduce the so-called Fichera function $F \colon \partial\Omega \to \mathbb{R}^N$ defined as

$$F(x) = \langle b(x), n(x)\rangle + \sum_{i,j=1}^{N} \partial_{x_j} a_{ij}(x) n_i(x).$$

It is named after Gaetano Fichera, who together with Olga Oleinik is the author of most of the results presented in this section. Using the Fichera function, we subdivide Σ^0 into the four parts given by

$$\Sigma_\pm = \{x \in \Sigma^0 \mid \pm F > 0\},\ \Sigma_0 = \{x \in \Sigma^0 \mid F = 0\} \text{ and } \Sigma_c = \partial\Omega \setminus \Sigma^0.$$

This partition is called the sigma partition of the boundary. If the boundary is only piecewise smooth, one needs to change the definition of the sigma-partition of the boundary slightly. In this case we are going to consider interior points of each of the smooth parts of the boundary only. It turns out that the right Cauchy problem to solve is

$$\begin{cases} Eu = h & \text{in } \Omega \\ u = f & \text{on } \Sigma_- \cup \Sigma_c \end{cases} \tag{2.3.2}$$

for suitable functions h and f.

Let us consider the case $E = -\Delta$, then $\Sigma^0 = \emptyset = \Sigma_-$ and thus $\Sigma_c = \partial\Omega$. In this case the Cauchy problem (2.3.2) is the classical Dirichlet problem for the Laplacian. We highlight the fact that it is not possible, at least using this theory, to prescribe boundary values on Σ_0 and Σ_+. For a physical interpretation on the sets Σ_- and Σ_c we refer to section 3.3.2. Let us anticipate that the set Σ_- describes in some sense the part of the boundary where an inflow takes place, while Σ_+ describes the boundary part where an outflow happens.

We note that every degenerate parabolic-elliptic second order partial differential equation

$$\partial_t u = \operatorname{tr}(A\nabla^2 u) + \langle b, \nabla u\rangle + cu$$

can also be stated in the form of equation (2.3.2). To see this, we introduce the parabolic cylinder $Q = (0, T) \times \Omega$ and define new coefficient functions as

$$\tilde{A}(x) = \begin{pmatrix} 0 & 0 \\ 0 & A(t,x) \end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+1)},$$

$\tilde{b} = (1, -b(t,x)) \in \mathbb{R}^{N+1}$ and $\tilde{c}(t,x) = -c(t,x)$ for all $(t,x) \in \mathbb{R}^N$. The boundary of the parabolic cylinder Q is only piecewise smooth. Let us therefore introduce the set

$$\partial_p Q = \{0\} \times \Omega \cup (0,T) \times \partial\Omega \cup \{T\} \times \Omega$$

containing the interior points of the smooth parts of the boundary of ∂Q. In the sense of equation (2.3.1) this leads to the partial differential equation

$$\partial_t u = \operatorname{tr}(A\nabla^2 u) + \langle b, \nabla u\rangle + cu + f.$$

Let us calculate the sigma-partition corresponding to the degenerate parabolic equation. We denote by $\Sigma_T^0, \Sigma_\pm^T$ and Σ_0^T the sigma sets to the elliptic problem with coefficients $\tilde{A}, \tilde{b}, \tilde{c}$ on the set Q. The sigma sets without the superscript denote the sigma partition corresponding to the elliptic problem with coefficients A, b, c on the set Ω. The outer unit normal $\tilde{n}$ to ∂Q in a point $(t,x) \in \partial_p Q$ is given as

$$\tilde{n}(t,x) = (-1, 0) \qquad\qquad \text{if } t = 0, x \in \Omega$$

$$\tilde{n}(t,x) = (0, n(x)) \qquad \text{if } t \in (0,T), x \in \partial\Omega$$
$$\tilde{n}(t,x) = (1, 0) \qquad \text{if } t = T, x \in \Omega,$$

where $n(x)$ denotes the outer unit normal to Ω in $x \in \partial\Omega$. First of all, it holds

$$\begin{aligned}\Sigma_T^0 &= \{(t,x) \in \partial_p Q \mid 0 = \left\langle \tilde{A}(x)\tilde{n}(x), \tilde{n}(x) \right\rangle = \langle A(x)n(x), n(x)\rangle\} \\ &= \{0\} \times \Omega \cup (0,T) \times \Sigma_0 \cup \{T\} \times \Omega.\end{aligned}$$

We calculate the parabolic Fichera function

$$F_T(t,x) = -\mathbb{1}_{\{0\}\times\Omega}(t,x) + \mathbb{1}_{\{T\}\times\Omega}(t,x) + F(x)\mathbb{1}_{(0,T)\times\partial\Omega}(t,x),$$

whence

$$\begin{aligned}\Sigma_+^T &= \{(t,x) \in \Sigma_T^0 \mid F_T(t,x) > 0\} = \{T\} \times \Omega \cup (0,T) \times \Sigma_+, \\ \Sigma_-^T &= \{(t,x) \in \Sigma_T^0 \mid F_T(t,x) < 0\} = \{0\} \times \Omega \cup (0,T) \times \Sigma_-\end{aligned}$$

and

$$\Sigma_c^T = \partial_p Q \setminus \Sigma_T^0 = (0,T) \times \Sigma_c.$$

Consequently, the homogenous Cauchy problem (2.3.2) with coefficients $\tilde{A}, \tilde{b}, \tilde{c}$ on the set Q is equivalent to

$$\begin{cases} \partial_t u = \operatorname{tr}(A\nabla^2 u) + \langle b, \nabla u\rangle + cu & t > 0, x \in \Omega \\ u(0,x) = f(0,x) & x \in \Omega \\ u(t,x) = f(t,x) & t \in (0,T), x \in \Sigma_- \cup \Sigma_c \end{cases}$$

where a suitable function f, which prescribes the initial datum and the boundary values for positive times on $\Sigma_- \cup \Sigma_c$, is given.

If $A = \mathrm{Id}_N$, $b = 0$ and $c = 0$, we end up with the Dirichlet problem for the classical heat equation on a bounded domain. Another enlightening example is the case where $A = 0$ and $c = 0$. In this situation we are looking at the transport equation corresponding to the vector field b. Let $\Omega \subset \mathbb{R}^N$, then the important sets of the sigma partition are given by

$$\begin{aligned}\Sigma_-^T &= \{0\} \times \Omega \cup (0,T) \times \{x \in \partial\Omega \mid \langle b, n\rangle < 0\}, \\ \Sigma_c^T &= \emptyset.\end{aligned}$$

Given a suitable function f, the Cauchy problem consists of finding a function such that

$$\begin{cases} \partial_t u = \langle b, \nabla u \rangle & t > 0, x \in \Omega \\ u(0,x) = f(0,x) & x \in \Omega \\ u(t,x) = f(t,x) & t \in (0,T), x \in \Sigma_-. \end{cases}$$

This means that we have to prescribe boundary values on every point of the boundary where the vector field points inward. In figure 2.1 one can see an example of such a situation. The

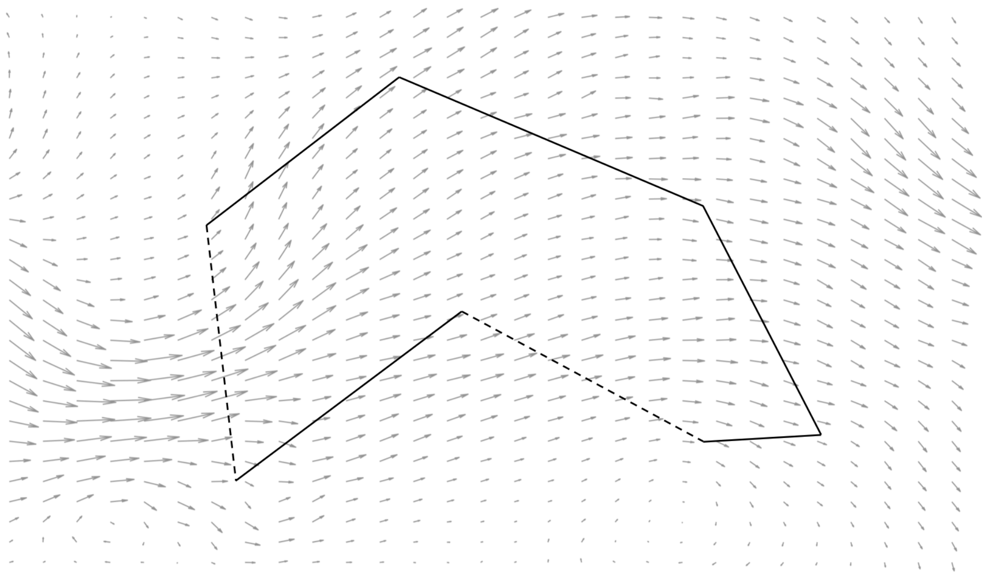

Figure 2.1: The partition of the boundary for some transport equation

dotted parts of the boundary correspond to the set Σ_-. We note that we are allowed to prescribe how much of the mass is transported into the domain but not how much mass leaves the domain. This makes sense, especially from a physical point of view.

2.3.2 A notion of weak solutions and an existence result

We want to define weak solutions to the Cauchy problem (2.3.2) and present a result on the existence of such. As in the previous section, we introduce the formal adjoint E^T of the differential operator E given by

$$E^T\varphi = -\operatorname{div}(A\nabla\varphi) - \langle b, \nabla\varphi \rangle - \sum_{i,j=1}^{N} \partial_{x_j} a_{ij} \partial_{x_i}\varphi - \sum_{i,j=1}^{N} \varphi \partial_{x_i}\partial_{x_j} a_{ij} + (c - \operatorname{div}(b))\varphi$$

for suitable functions φ to provide a shorthand way of writing

$$\langle h, \varphi \rangle = \left\langle u, E^T \varphi, \right\rangle$$

where u is a smooth solution to the Cauchy problem (2.3.2), where $h \in L^p(\Omega)$ and $\varphi \in C^\infty(\overline{\Omega})$ with zero trace on $\Sigma_- \cup \Sigma_c$. This is how we define weak solutions to the Cauchy problem in $L^p(\Omega)$. We are going to consider the case of homogenous boundary conditions, i.e. $f = 0$.

Definition 2.3.1. We say that a function $u \in L^p(\Omega)$ is a weak solution of the Cauchy problem (2.3.2), where $h \in L^p(\mathbb{R}^N)$, if for all $\varphi \in C^\infty(\Omega)$ such that $\varphi|_{\Sigma_- \cup \Sigma_c} = 0$, it holds

$$\langle h, \varphi \rangle = \left\langle u, E^T \varphi \right\rangle .$$

Theorem 2.3.2. *Let $p \in (1, \infty)$. We assume that $c > 0$ on $\overline{\Omega}$ and that*

$$-\mathrm{div}(b) - \sum_{i,j=1}^{N} \partial_{x_i} \partial_{x_j} a_{ij} + c > 0$$

on $\overline{\Omega}$. For every $h \in L^p(\Omega)$ there exists a weak solution to (2.3.2) *in the sense of definition 2.3.1. If $p \geq 3$, the weak solution is unique.*

Proof. [WYW06, Theorem 13.1.2] and [Ole73, Chapter 1]. □

Remark 2.3.3. (i) If $1 \leq p < 3$, then the weak solution to (2.3.2) is in general not unique. For further information we refer to [Ole73, Chapter 1].

(ii) Note that this is a different notion of weak solution than usual. There are also existence results for weak solution in terms of weak derivatives. But for such existence results one needs more assumptions on the domain Ω. A prominent theorem, which is due to Kohn and Nirenberg, gives existence of weak solutions in the Sobolev sense under additional assumptions. This can be found in [KN67] or in [Ole73, Section 1.9].

(iii) Let us note that there is an extension of this theory that allows also Neumann type boundary conditions. More information on this matter can be found in [Fic59].

2.4 Hörmanders theory of hypoelliptic operators

2.4.1 Partial differential operators and Lie algebras

Definition 2.4.1. We consider partial differential operators of the form

$$P = \sum_{|\alpha| \leq m} c_\alpha(x) \partial^\alpha$$

with coefficients $c_\alpha \colon \Omega \to \mathbb{R}$. An operator P of above shape is said to be of order m. We say that P is constant if all coefficients c_α are constant functions in Ω and we call P homogeneous if $c_0 = 0$.

Definition 2.4.2. A partial differential operator is called hypoelliptic if for every open subset $\Omega \subset \mathbb{R}^n$ and every distribution $u \in \mathcal{D}'(\Omega)$ such that $Pu \in C^\infty(\Omega)$, it holds that $u \in C^\infty(\Omega)$, i.e. the distribution u can be represented by a function $u \in C^\infty(\Omega)$.

Remark 2.4.3. Every elliptic second order partial differential operator is hypoelliptic. This is for example shown in [Shi92, Chapter 4, Theorem 2.1].

Definition 2.4.4. A Lie algebra is a vector space V over a field $\mathbb{K}$ equipped with a so-called Lie bracket $[\cdot,\cdot] \colon V \times V \to V$, $(x,y) \mapsto [x,y]$. A Lie bracket is a bilinear map which satisfies the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[y,z]] = 0$$

for all $x, y, z \in V$ as well as $[x,x] = 0$ for all $x \in V$. The dimension of a Lie algebra is the dimension of the corresponding vector space.

Example 2.4.5. We consider the space $C^\infty(\mathbb{R}^N; \mathbb{R}^N)$ of all smooth vector fields. Given two vector fields X, Y, we define

$$[X,Y](x) = DV(x)W(x) - DW(x)V(x)$$

for all $x \in \mathbb{R}^n$. Then $(C^\infty(\mathbb{R}^N; \mathbb{R}^N), [\cdot,\cdot])$ is a Lie algebra. Let $X = \sum_{i=1}^n a_i(x) \partial_{x_i}$ be a smooth first order differential operator on functions $u \colon \mathbb{R}^n \to \mathbb{R}$. We might interpret this differential operator as a vector field by viewing ∂_{x_i} as the i-th unit vector. By this identification we might regard the first order differential operators as a Lie algebra.

Definition 2.4.6. Consider a Lie algebra $(V, [\cdot,\cdot])$. Given vectors $v_1, \dots, v_n$, we denote by $\mathrm{Lie}(v_1, \dots, v_n)$ the smallest subspace of V such that $\mathrm{Lie}(v_1, \dots, v_n)$ endowed with the restriction of $[\cdot,\cdot]$ is a Lie algebra.

2.4.2 Hörmander's theorem

In this section we are going to consider general second order differential operators P that can be written as a sum of squares of first order differential operators. Such an operator P is of the form

$$P = \sum_{j=1}^{r} X_j^2 + X_0 + c, \tag{2.4.1}$$

with real C^∞ coefficients on an open set $\Omega \subset \mathbb{R}^N$. $X_0, \ldots, X_r$ denote first order homogenous differential operators in the open set $\Omega \subset \mathbb{R}^N$ with C^∞ coefficients and we assume that $c \in C^\infty(\Omega)$. The following theorem is due to Lars Hörmander. It is inspired by the work of Andrei Kolmogorov on the Kolmogorov equation. We are going to investigate the connection between the following theorem and the Kolmogorov equation in chapter 6.

Theorem 2.4.7. *Let P be a second order differential operator as in equation* (2.4.1) *and assume that for every $x \in \Omega$ it holds that*

$$\dim \operatorname{Lie}\left(\{X_{j_1}, [X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}], \ldots \mid j_i = 0, \ldots, n\}\right) = n, \tag{2.4.2}$$

then the operator P is hypoelliptic.

Proof. [Hör67, Theorem 1.1] □

3 The Cauchy problem for Kolmogorov equations

In this chapter we are going to study the degenerate parabolic Cauchy problem for equations similar to the Kolmogorov equation:

$$\partial_t u + v \cdot \nabla_x u = \Delta_v u.$$

We start by investigating equations with constant coefficients on $\mathbb{R}^N$. Afterwards, we are going to apply the results from section 2.2 to Kolmogorov equations with variable diffusion coefficient. At the end of the chapter we are going to study Kolmogorov equations with constant coefficients on bounded domains.

3.1 Kolmogorov equations with constant coefficients

In this section we are going to study the well-posedness of Kolmogorov equations with constant coefficients. To be more precise, given the two matrices $A, B \in \mathbb{R}^{N\times N}$, we want to study the linear partial differential operator

$$\mathcal{K}u = \operatorname{div}(A\nabla u) + \langle x, B\nabla u\rangle$$

on suitable functions $u \colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. Moreover, we are interested in the degenerate parabolic problem

$$\partial_t u = \mathcal{K}u = \operatorname{div}(A\nabla u) + \langle x, B\nabla u\rangle \,. \tag{3.1.1}$$

We make the following structural assumptions on the matrices A, B. Let $m_0, \ldots, m_r$ be natural numbers such that $m_0 \geq m_1 \geq \cdots \geq m_r \geq 1$ and $\sum_{k=0}^{r} m_k = N$. We assume the diffusion matrix A to be of the form

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$$

for a symmetric and positive definite matrix $A_0 \in \mathbb{R}^{m_0 \times m_0}$. B will be given by

$$B = \begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_r \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with matrices $B_k \in \mathbb{R}^{m_{k-1} \times m_k}$ of rank m_k for $k = 1, \ldots, r$. At this stage we want to fix some useful notation. The dilation group on $\mathbb{R}^N$ is described by the matrix $\delta_\lambda \in \mathbb{R}^{N \times N}$, given as

$$\delta_\lambda = \operatorname{diag}(\lambda \operatorname{Id}_{m_0}, \lambda^3 \operatorname{Id}_{m_1}, \ldots, \lambda^{2r+1} \operatorname{Id}_{m_r})$$

for $\lambda > 0$. The number $Q = m_0 + 3m_1 + \cdots + (2r+1)m_r$ is called the homogenous dimension of $\mathbb{R}^N$. It holds $\det(\delta_\lambda) = \lambda^Q$. An arbitrary vector $x \in \mathbb{R}^N$ can be written as $x = (x^{(0)}, \ldots, x^{(r)})$ with $x^{(k)} \in \mathbb{R}^{m_k}$ for all $k = 0, \ldots, r$. The first order term of the degenerate parabolic problem (3.1.1) will be denoted by Y, that is $Y = \langle x, B\nabla \rangle - \partial_t$. The structural assumptions made on $\mathcal{K}$ will be assumed throughout the complete section 3.1.

Example 3.1.1. We choose $N = 2n$, $m_0 = m_1 = n$, $A_0 = \operatorname{Id}_n$ and $B_1 = -\operatorname{Id}_n$ for some $n \in \mathbb{N}$. In this case $\mathcal{K}$ is the differential operator corresponding to the Kolmogorov equation

$$\partial_t u + v \cdot \nabla_x u = \Delta_v u. \tag{3.1.2}$$

As suggested in the introduction, the Kolmogorov equation is a forward Kolmogorov equation to a stochastic differential equation. To be more precise, this partial differential equation is the forward Kolmogorov equation corresponding to the following Itô stochastic differential equation:

$$\begin{aligned} \mathrm{d}X^i(t) &= V(t)^i \mathrm{d}t && \text{for } i = 1, \ldots, n, \\ \mathrm{d}V^i(t) &= \sqrt{2}\mathrm{d}W^i(t) && \text{for } i = 1, \ldots, n \end{aligned}$$

for an n-dimensional Wiener process $W(t)$. This system describes the evolution of a particle at the position X with velocity V, driven by a Brownian fluctuation of its velocity V. For more information on stochastic differential equations and their forward Kolmogorov equations we refer to the book [SV06].

Lemma 3.1.2. The differential operator $\mathcal{K}$ is homogenous with respect to the dilations group δ_λ on $\mathbb{R}^N$ that is

$$\mathcal{K}(u(\delta_\lambda x))(x) = \lambda^2 [\mathcal{K}u](\delta_\lambda x)$$

for all suitable functions u, any $\lambda > 0$ and every $x \in \mathbb{R}^N$.

Proof. It holds

$$\begin{aligned}\mathcal{K}(u(\delta_\lambda x)) &= \lambda^2 \mathrm{div}(A[\nabla u](\delta_\lambda x)) + \left\langle \delta_\lambda x, \delta_{\frac{1}{\lambda}} B \delta_\lambda [\nabla u](\delta_\lambda x) \right\rangle \\ &= \lambda^2 [\mathcal{K} u](\delta_\lambda x),\end{aligned}$$

since $A\delta_{\lambda^2} = \lambda^2 A$ and $\delta_{\frac{1}{\lambda}} B \delta_\lambda = \lambda^2 B$. □

Remark 3.1.3. (i) Under the above structural assumptions on A and B the operator $\mathcal{K} - \partial_t$ satisfies the Hörmander rank condition. This will be investigated in chapter 6.1.

(ii) It turns out that every linear partial differential operator $\mathcal{K} = \mathrm{div}(A\nabla) + \langle x, B\nabla \rangle$ for arbitrary matrices $A, B \in \mathbb{R}^{N \times N}$, that is assumed to be homogenous with respect to the dilations group δ_λ, satisfies the above structural conditions in some sense. More information and a proof of this statement can be found in [LP94, Section 2].

(iii) $\mathcal{K}$ can also be seen as the partial differential operator corresponding to an Ornstein-Uhlenbeck process, i.e. if A is a symmetric positive definite and B an arbitrary matrix. These operators are somehow related to the Kolmogorov equation. In [Lor17] these non-degenerate partial differential equations are studied.

(iv) The results presented in this section are based on the articles [LP94] and [Pol95]. Some arguments are also inspired of the proofs for similar results for the heat equation as can be found for example in [Eva10, Section 2.3]. Finally, some arguments are based on [Lor17, Chapter 10].

3.1.1 The fundamental solution

We are going to derive a formula for the fundamental solution of (3.1.1). To simplify the presentation of the fundamental solution, we introduce some notation.

Definition 3.1.4. We define the matrix-valued functions $\mathcal{C}, E \colon \mathbb{R} \to \mathbb{R}^{N \times N}$ by

$$\mathcal{C}(t) = \int_0^t E(s) A E^T(s) \mathrm{d}s$$

and $E(t) = \exp(-tB^T)$ for all $t \in \mathbb{R}$.

We are going to discuss some important properties of these matrix-valued functions. We are going to write $S > T$ ($S \geq T$) for some matrices $S, T \in \mathbb{R}^N$ if $S - T$ is positive (semi-)definite.

Lemma 3.1.5. For all $t > 0$ it holds that $\mathcal{C}(t) > 0$.

Proof. Since A is positive semidefinite, we immediately conclude that the mapping $t \mapsto \langle \mathcal{C}(t)x, x\rangle$ is a monotone non-decreasing function for every $x \in \mathbb{R}^N$. We suppose that there is a $t > 0$ and $x \in \mathbb{R}^N$ such that $\langle \mathcal{C}(t)x, x\rangle = 0$. The monotony implies that $\langle \mathcal{C}(s)x, x\rangle = 0$ for all $0 \le s \le t$. Since the integrand in the definition of $\mathcal{C}(t)$ is nonnegative, we conclude $\langle A\exp(-tB)x, \exp(-tB)x\rangle = 0$ for all $0 \le s \le t$. Thus, $A\exp(-tB)x = 0$ for all $0 \le s \le t$. By definition of the matrix exponential function, we see that

$$\left(\sum_{k=0}^{\infty}(-1)^k \frac{s^k}{k!} AB^k\right) x = 0 \tag{3.1.3}$$

for all $0 \le s \le t$. This shows that $AB^k x = 0$ for all $k \in \mathbb{N}$ by equating coefficients. It is $\mathcal{N}(A) = \{x \in \mathbb{R}^N \mid x^{(0)} = 0\}$ and from straightforward matrix multiplication we deduce $A_0 B_1 \cdots B_k x^{(k)} = 0$ for all $k = 1, \dots, r$. Since B_k is of rank k and A_0 is symmetric positive definite, we iteratively deduce $x^{(k)} = 0$ for all $k = 1, \dots, r$ and thus $x = 0$. Consequently, $C(t)$ is positive definite for all $t > 0$. ☐

Let us now derive the fundamental solution. While the following calculations will only be formal, we will later see that the formula obtained is indeed meaningful. Let u be a nice solution of equation 3.1.1. We want to eliminate the drift term $\langle x, B\nabla u\rangle$ in equation (3.1.1). To do so, we define the function $v(t,x) = u(t, \exp(-tB^T)x) = u(t, E(t)x)$ and calculate the corresponding differential equation for v. It holds

$$\begin{aligned}\partial_t v(t,x) &= [\partial_t u](t, \exp(-tB^T)x) - \big\langle [\nabla u](t, \exp(-tB^T)x), B^T \exp(-tB^T)x\big\rangle \\ &= [\partial_t u](t, \exp(-tB^T)x) - \big\langle \exp(-tB^T)x, B[\nabla u](t, \exp(-tB^T)x)\big\rangle\end{aligned}$$

as well as

$$\partial_{x_i} v(t,x) = \sum_{s=1}^{N} [\partial_{x_s} u](t, \exp(-tB^T)x)\exp(-tB^T)_{si}$$

and

$$\partial_{x_l}\partial_{x_i} v(t,x) = \sum_{s,t=1}^{N} [\partial_{x_t}\partial_{x_s} u](t, \exp(-tB^T)x)\exp(-tB^T)_{si}\exp(-tB^T)_{tl}$$

for all $i, l = 1, \dots, N$. Using the latter calculation and that $\exp(-tB)^T = \exp(-tB^T)$, we obtain

$$\begin{aligned}&\operatorname{tr}\left(\exp(tB^T)A\exp(tB)\nabla^2 v(t,x)\right) \\ &= \sum_{i,j,k,l=1}^{N} \exp(tB^T)_{ij}A_{jk}\exp(tB)_{kl}\left(\nabla^2 v(t, \exp(-tB^T)x)\right)_{li}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j,k,l=1}^{N} \sum_{s,t=1}^{N} \exp(tB^T)_{ij} A_{jk} \exp(tB^T)_{kl} \left[\partial_{x_t}\partial_{x_s} u\right](t, \exp(-tB^T)x) \exp(-tB^T)_{tl} \exp(-tB^T)_{si} \\
&= \sum_{i,j,k,l=1}^{N} \sum_{s,t=1}^{N} \exp(-tB^T)_{si} \exp(tB^T)_{ij} A_{jk} \exp(tB)_{kl} \exp(-tB)_{lt} \left[\partial_{x_t}\partial_{x_s} u\right](t, \exp(-tB^T)x) \\
&= \operatorname{tr}\left(\exp(-tB^T)\exp(tB^T) A \exp(tB)\exp(-tB)[\nabla^2 u](t, \exp(-tB^T)x)\right) \\
&= \operatorname{tr}\left(A[\nabla^2 u]\right)(t, \exp(-tB^T)x) \\
&= [\operatorname{div}(A\nabla u)](t, \exp(-tB^T)x).
\end{aligned}
$$

Piecing these equations together and using equation (3.1.1), we conclude

$$
\begin{aligned}
\partial_t v(t,x) &= [\partial_t u](t, \exp(-tB^T)x) - \left\langle \exp(-tB^T)x, B[\nabla u](t, \exp(-tB^T)x)\right\rangle \\
&= [\operatorname{div}(A\nabla u)](t, \exp(-tB^T)x) + \left\langle \exp(-tB^T)x, B[\nabla u](t, \exp(-tB^T)x)\right\rangle \\
&\quad - \left\langle \exp(-tB^T)x, B[\nabla u](t, \exp(-tB^T)x)\right\rangle \\
&= \operatorname{tr}(\exp(tB^T) A \exp(tB) \nabla^2 v(t,x)).
\end{aligned}
$$

This gives the following partial differential equation for v

$$
\begin{cases}
\partial_t v = \operatorname{tr}(\exp(tB^T) A \exp(tB)\nabla^2 v(t,x)), & t > 0, x \in \mathbb{R}^n \\
v(0,x) = f(x), & x \in \mathbb{R}^n.
\end{cases}
$$

Using Fourier transformation in the space variable on both sides of the differential equation, we obtain the following differential equation for the Fourier transform $\hat{v}$ of v:

$$
\begin{cases}
\partial_t \hat{v}(t,\xi) = -\left\langle \exp(tB^T) A \exp(tB)\xi, \xi\right\rangle \hat{v}(t,\xi), & t > 0, \xi \in \mathbb{R}^n \\
\hat{v}(0,\xi) = \hat{f}(\xi), & \xi \in \mathbb{R}^n.
\end{cases}
$$

Making use of separation of variables, we see that this equation is solved by

$$
\hat{v}(t,\xi) = \exp\left(-\left\langle C(t)\xi, \xi\right\rangle\right) \hat{f}(\xi)
$$

where $C(t) = \int_0^t \exp(sB^T) A \exp(sB) \mathrm{d}s$. By a change of variable in the integrand, we see that

$$
\exp(tB) C(t)^{-1} \exp(tB^T) = \mathcal{C}(t)^{-1}. \tag{3.1.4}
$$

We recall that multiplication in the Fourier variable becomes convolution in the physical variable. Therefore, we calculate

$$
\mathcal{F}^{-1}\left(\xi \mapsto \exp\left(-\left\langle C(t)\xi, \xi\right\rangle\right)\right)(x)
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{(2\pi)^N}} \int_{\mathbb{R}^N} \exp\left(i\left\langle \xi, x\right\rangle\right) \exp\left(-\left\langle C(t)\xi, \xi\right\rangle\right) \mathrm{d}\xi \\
&= \frac{1}{\sqrt{(4\pi)^N \det C(t)}} \int_{\mathbb{R}^N} \exp\left(i\left\langle C(t)^{-\frac{1}{2}} \frac{x}{\sqrt{2}}, y\right\rangle\right) \exp\left(-\frac{1}{2}|y|^2\right) \mathrm{d}y \\
&= \frac{1}{\sqrt{(2\pi)^N \det C(t)}} \exp\left(-\frac{1}{4}\left\langle C(t)^{-\frac{1}{2}}x, C(t)^{-\frac{1}{2}}x\right\rangle\right) \\
&= \frac{1}{\sqrt{(2\pi)^N \det C(t)}} \exp\left(-\frac{1}{4}\left\langle C(t)^{-1}x, x\right\rangle\right),
\end{aligned}
$$

where we write $C(t)^{-\frac{1}{2}}$ for the inverse of the square root of $C(t)$. Here we have used the fact that $C(t)$ is symmetric positive definite as seen by lemma 3.1.5 and equation (3.1.4). Moreover we have used that $\exp\left(-\frac{1}{2}|y|^2\right)$ is a fixed point of the Fourier transform. We conclude that

$$
\begin{aligned}
v(t,x) &= \mathcal{F}^{-1}(\hat{v}(t,\cdot))(x) \\
&= \frac{1}{\sqrt{(2\pi)^N}} \mathcal{F}^{-1}\left(\xi \mapsto \exp\left(-\left\langle C(t)\xi, \xi\right\rangle\right)\right) * f \\
&= \frac{1}{\sqrt{(4\pi)^N \det C(t)}} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{4}\left\langle C(t)^{-1}(x-y), x-y\right\rangle\right) f(y)\mathrm{d}y.
\end{aligned}
$$

Since $v(t,x) = u(t, \exp(-tB^T)x)$, we deduce

$$
u(t,x) = \frac{1}{\sqrt{(4\pi)^N \det \mathcal{C}(t)}} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{4}\left\langle \mathcal{C}(t)^{-1}\left(x - E(t)y\right), x - E(t)y\right\rangle\right) f(y)\mathrm{d}y,
$$

using equation (3.1.4) as well as $\det\left(\exp(-tB^T)\right) = 1$.

Definition 3.1.6. The fundamental solution of $\partial_t u = \mathcal{K}u$ with pole at 0 is defined as

$$
\begin{aligned}
&\Gamma(t_2, x_2, t_1, x_1) \\
&= \left((4\pi)^N \det \mathcal{C}(t_2 - t_1)\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{4}\left\langle \mathcal{C}^{-1}(t_2-t_1)(x_2 - E(t_2-t_1)x_1), x_2 - E(t_2-t_1)x_1\right\rangle\right)
\end{aligned}
$$

for $t_2 > t_1$ and $\Gamma(t_2, x_2, t_1, x_1) = 0$ for $t_2 \leq t_1$. We are going to write $\Gamma(t,x) = \Gamma(t,x,0,0)$ for every $(t,x) \in \mathbb{R}^{N+1}$.

Remark 3.1.7. Choosing $f = \delta_0$, the Dirac measure in $x = 0$, we see that for all $t > 0$ it holds

$$
\partial_t \Gamma(t,x) = \operatorname{div}(A\nabla\Gamma(t,x)) + \left\langle x, B\nabla\Gamma(t,x)\right\rangle.
$$

To make this argument rigorous, we note that choosing $\hat{f} = 1$ in above calculation, the following arguments become immediately valid for $t > 0$ and thus $\Gamma(t,x)$ indeed solves the given partial differential equation in $(0,\infty)\times\mathbb{R}^N$. Noting that $\Gamma(t,x,s,y) = \Gamma(t-s, x-E(t-s)y)$, we calculate

$$\partial_t\Gamma(t-s,x-E(t-s)y) = [\partial_t\Gamma](t-s,x-E(t-s)y) - \langle E(t-s)y, B[\nabla\Gamma](t-s,x-E(t-s)y)\rangle$$

and therefore deduce

$$\partial_t\Gamma(t,x,s,y) = \operatorname{div}(A\nabla_x\Gamma(t,x,s,y)) + \langle x, B\nabla_x\Gamma(t,x,s,y)\rangle \tag{3.1.5}$$

for all $t > s > 0$. The same calculation for arbitrary functions u also shows the invariance of $\mathcal{K} - \partial_t = 0$ with respect to the left translation of the group $(t,x)\circ(s,y) = (t+s, y+E(s)x)$.

In the following part of this subsection we collect some interesting properties of the fundamental solution. These properties are going to be important throughout the remaining chapters.

Lemma 3.1.8. Let $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, then

$$E(t) = \sum_{k=0}^{r}(-1)^k\frac{t^k}{k!}\left(B^T\right)^k \tag{3.1.6}$$

and thus

$$|E(t)x| \le t^r c(B)\,|x|$$

for all $t > 1$ and a constant $c(B) > 0$. Moreover, it is

$$\left|\delta_{\frac{1}{\sqrt{t}}}E(t)y\right| \le t^{-\frac{1}{2}}c(B)\,|y|$$

for all $t > 1$ and any $y \in \mathbb{R}^N$.

Proof. Note that B is nilpotent of order $r+1$ and thus B^T is nilpotent, too. This shows equation (3.1.6). The first estimate follows by the triangle inequality. For the second inequality we note that multiplying a matrix from the left with a diagonal matrix is equivalent to multiplying all rows by the diagonal entries. Due to the structure of B, the highest order of t appearing in $\delta_{\frac{1}{\sqrt{t}}}(B^T)^k$ is $t^{-\frac{2k+1}{2}}$. Therefore, $t^k\delta_{\frac{1}{\sqrt{t}}}(B^T)^k$ is of order $t^{-\frac{1}{2}}$. To visualize this, we consider the case of $r = 2$, $N = 3$, $m_0 = m_1 = m_2 = 1$ and $B_1, B_2 \neq 0$. It holds

$$\delta_{\frac{1}{\sqrt{t}}}B^T = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{t}^3}B_1^T & 0 & 0 \\ 0 & \frac{1}{\sqrt{t}^5}B_2^T & 0 \end{pmatrix} \text{ and } \delta_{\frac{1}{\sqrt{t}}}(B^T)^2 = \frac{1}{\sqrt{t}^5}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B_2^T B_1^T & 0 & 0 \end{pmatrix}.$$

The general case follows by straightforward matrix multiplication. □

Lemma 3.1.9. The matrix $\mathcal{C}(t)$ satisfies the equation $\mathcal{C}(t) = \delta_{\sqrt{t}}\mathcal{C}(1)\delta_{\sqrt{t}}$ for all $t > 0$.

Proof. Using equation (3.1.6), we first show that it holds $E(\lambda^2 t) = \delta_\lambda E(t)\delta_{\frac{1}{\lambda}}$ for all $\lambda, t > 0$. It is

$$\begin{aligned}\delta_\lambda E(t)\delta_{\frac{1}{\lambda}} &= \sum_{k=0}^{r}(-1)^k\frac{t^k}{k!}\delta_\lambda\left(B^T\right)^k\delta_{\frac{1}{\lambda}} = \sum_{k=0}^{r}(-1)^k\frac{t^k}{k!}\left(\delta_\lambda B^T\delta_{\lambda^{-1}}\right)^k \\ &= \sum_{k=0}^{r}(-1)^k\frac{t^k}{k!}\left(\lambda^2 B^T\right)^k = E(\lambda^2 t),\end{aligned}$$

since due to the structural assumption made on B, we have $\delta_\lambda B^T\delta_{\lambda^{-1}} = \lambda^2 B^T$. Let $t > 0$, then it holds

$$\begin{aligned}\int_0^t E(s)AE(s)^T \mathrm{d}s = t\int_0^1 E(st)AE(st)^T\mathrm{d}s &= t\int_0^1 \delta_{\sqrt{t}}E(s)\delta_{\frac{1}{\sqrt{t}}}A\delta_{\frac{1}{\sqrt{t}}}E(s)^T\delta_{\sqrt{t}}\mathrm{d}s \\ &= \delta_{\sqrt{t}}\mathcal{C}(1)\delta_{\sqrt{t}}\end{aligned}$$

according to the identity $\delta_{\frac{1}{\sqrt{t}}}A\delta_{\frac{1}{\sqrt{t}}} = \frac{1}{t}A$. □

Corollary 3.1.10. For every $\lambda, t > 0$ it holds $E(\lambda^2 t) = \delta_\lambda E(t)\delta_{\frac{1}{\lambda}}$.

Corollary 3.1.11. For all $t > 0$ the determinant of $\mathcal{C}(t)$ is given as $\det\mathcal{C}(t) = t^Q\det\mathcal{C}(1)$.

Corollary 3.1.12. Introducing the constant $c_0 = \left((4\pi)^N\det\mathcal{C}(1)\right)^{-\frac{1}{2}}$, the fundamental solution of $\partial_t u = \mathcal{K}u$ with pole at zero can also be written as

$$\begin{aligned}&\Gamma(t_2, x_2, t_1, x_1) \\ &= \frac{c_0}{(t_2-t_1)^{\frac{Q}{2}}}\exp\left(-\frac{1}{4}\left\langle\mathcal{C}^{-1}(t_2-t_1)(x_2-E(t_2-t_1)x_1), x_2-E(t_2-t_1)x_1\right\rangle\right)\end{aligned}$$

or as

$$\begin{aligned}&\Gamma(t_2, x_2, t_1, x_1) \\ &= \frac{c_0}{(t_2-t_1)^{\frac{Q}{2}}}\exp\left(-\frac{1}{4}\left\langle\mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t_2-t_1}}}(x_2-E(t_2-t_1)x_1), \delta_{\frac{1}{\sqrt{t_2-t_1}}}(x_2-E(t_2-t_1)x_1)\right\rangle\right)\end{aligned}$$

for $t_2 > t_1$ and $\Gamma(t_2, x_2, t_1, x_1) = 0$ for $t_2 \leq t_1$.

Example 3.1.13. We want to calculate the fundamental solution of the Kolmogorov equation

$$\partial_t u + v\cdot\nabla_x u = \Delta_v u. \tag{3.1.7}$$

Note that in this case B is nilpotent of order 2 so that we have

$$E(t) = \exp(-tB^T) = \mathrm{Id}_N - tB^T = \begin{pmatrix} \mathrm{Id}_n & 0 \\ t\,\mathrm{Id}_n & \mathrm{Id}_n \end{pmatrix}. \tag{3.1.8}$$

A straightforward calculation shows that

$$\mathcal{C}(t) = \frac{1}{6}\begin{pmatrix} 6t\,\mathrm{Id}_n & 3t^2\,\mathrm{Id}_n \\ 3t^2\,\mathrm{Id}_n & 2t^3\,\mathrm{Id}_n \end{pmatrix}$$

for all $t \in \mathbb{R}$. From this we calculate $\det(\mathcal{C}(1)) = 12^{-n}$. Making the ansatz that the inverse of $\mathcal{C}(t)$ is once again a block matrix, we deduce

$$\mathcal{C}^{-1}(t) = \frac{1}{t^3}\begin{pmatrix} 4t^2\,\mathrm{Id}_n & -6t\,\mathrm{Id}_n \\ -6t\,\mathrm{Id}_n & 12\,\mathrm{Id}_n \end{pmatrix}.$$

Let $x \in \mathbb{R}^N$ and $t > 0$, then

$$\Gamma(t,x) = \frac{\sqrt{3}^n}{(2\pi)^n t^{2n}} \exp\left(-\frac{1}{t}\left|x^{(0)}\right|^2 + \frac{3}{t^2}\left\langle x^{(0)}, x^{(1)}\right\rangle - \frac{3}{t^3}\left|x^{(1)}\right|^2\right). \tag{3.1.9}$$

Let us compare this expression with the one Andrej Kolmogorov obtained in his paper [Kol34]. He stated the fundamental solution in the case $n = 1$ and $(v,x) \in \mathbb{R}^2$ as

$$\widetilde{\Gamma}(t,x) = \frac{2\sqrt{3}}{\pi t^2}\exp\left(-\frac{v^2}{4t} - \frac{3\left(x+\frac{v}{2}\right)^2}{t^3}\right) = \frac{2\sqrt{3}}{\pi t^2}\exp\left(-\frac{v^2}{t} + \frac{3xv}{t^2} - \frac{3x^2}{t^3}\right).$$

We see that his formula actually coincides with the one presented here up to a multiplicative normalization constant.

Lemma 3.1.14. For all $t > s \geq 0$ and every $x \in \mathbb{R}^N$ it holds

$$\int_{\mathbb{R}^N} \Gamma(t,x,s,y)\mathrm{d}y = 1.$$

Proof. It holds

$$\begin{aligned}\int_{\mathbb{R}^N} \Gamma(t,x,s,y)\mathrm{d}y &= \frac{c_0}{(t-s)^{\frac{Q}{2}}}\int_{\mathbb{R}^N}\exp\left(-\frac{1}{4}\left\langle \mathcal{C}^{-1}(t-s)(x-E(t-s)y), x-E(t-s)y\right\rangle\right)\mathrm{d}y \\ &= \frac{c_0}{(t-s)^{\frac{Q}{2}}}\int_{\mathbb{R}^N}\exp\left(-\frac{1}{4}\left\langle \mathcal{C}^{-1}(t-s)z, z\right\rangle\right)\mathrm{d}z,\end{aligned}$$

since $\det(E(t-s)) = e^{\mathrm{tr}((t-s)B^T)} = 1$. Therefore, we conclude by lemma 3.1.5 together with lemma B.0.5

$$\int_{\mathbb{R}^N} \Gamma(t,x,s,y)\mathrm{d}y = \frac{c_0}{(t-s)^{\frac{Q}{2}}} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{4}\left\langle \mathcal{C}^{-1}(t-s)z, z\right\rangle\right)\mathrm{d}z$$
$$= \frac{c_0}{(t-s)^{\frac{Q}{2}}}\sqrt{\det(4\pi\mathcal{C}(t-s))} = 1$$

by corollary 3.1.11. □

Lemma 3.1.15. For every $t > s > 0$ and any $x, y \in \mathbb{R}^N$ it holds $\Gamma(t,x,s,y) > 0$.

Proof. This follows immediately from the positivity of the exponential function. □

Theorem 3.1.16. *For every sequence $t_k \to 0$ it holds that the sequence $(\Gamma(t_k,\cdot,0,\cdot))_{k\in\mathbb{N}}$ is a generalized Dirac sequence. In particular, the sequence $(\Gamma(t_k,\cdot,0,0))_{k\in\mathbb{N}}$ is a Dirac sequence.*

Proof. It follows from lemma 3.1.15 that this sequence consists of nonnegative functions. Moreover, by lemma 3.1.14, each function is integrable in the second argument with integral equal to one. It remains to validate condition (iii)' of definition A.3.1. Let $\delta > 0$, $x \in \mathbb{R}^N$, $x_k \to x$ and $K \in \mathbb{N}$ such that $|x_k - x| < \frac{\delta}{2}$ for all $k \geq K$, then

$$\int_{\mathbb{R}^N\setminus B_\delta(x)} \Gamma(t_k,x_k,0,y)\mathrm{d}y$$
$$= \frac{c_0}{t_k^{\frac{Q}{2}}} \int_{\mathbb{R}^N\setminus B_\delta(x)} \exp\left(-\frac{1}{4}\left\langle \mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t_k}}}(x_k - E(t_k)y), \delta_{\frac{1}{\sqrt{t_k}}}(x_k - E(t_k)y)\right\rangle\right)\mathrm{d}y$$
$$\leq \frac{c_0}{t^{\frac{Q}{2}}} \int_{\mathbb{R}^N\setminus B_\delta(x)} \exp\left(-\frac{1}{4}\lambda_1\left\langle \delta_{\frac{1}{\sqrt{t_k}}}(x_k - E(t_k)y), \delta_{\frac{1}{\sqrt{t_k}}}(x_k - E(t_k)y)\right\rangle\right)\mathrm{d}y$$

for a constant $\lambda_1 > 0$, since $\mathcal{C}^{-1}(1)$ is positive definite.

To estimate further, we need to control the term in the exponential function. We let $t_0 \in (0,1)$ such that $|x_k - E(-t_k)x_k| < \frac{\delta}{6}$ for all $t_k \in (-t_0, t_0)$ and any $k \geq K$. Such a t_0 exists, since the map $\mathbb{R}^{N+1} \to \mathbb{R}^N$, $(t,x) \mapsto E(t)x$ is uniformly continuous on compact subsets of $\mathbb{R}^N$. If $|y - x| \geq \delta$, $|x - x_k| \leq \frac{\delta}{2}$ and $t_k \in (0, t_0)$, we deduce

$$|y-x| \leq |y - E(-t_k)x_k| + |E(-t_k)x_k - x_k| + |x_k - x| \leq \frac{2\delta}{3} + |y - E(-t_k)x_k|$$
$$\leq \frac{2}{3}|y-x| + |y - E(-t_k)x_k|$$

and conclude

$$-|y - E(-t_k)x_k|^2 \leq -\frac{1}{9}|y-x|^2.$$

In particular, there is a $\tilde{K} \geq K$ such that $t_k < t_0$ and

$$- |y - E(-t_k)x_k|^2 \leq -\frac{1}{9} |y - x|^2$$

for all $k \geq \tilde{K}$. Using corollary 3.1.10 with $\lambda = \frac{1}{\sqrt{t_k}}$, we deduce

$$\begin{aligned}
&\left\langle \delta_{\frac{1}{\sqrt{t_k}}}(x_k - E(t_k)y), \delta_{\frac{1}{\sqrt{t_k}}}(x_k - E(t_k)y) \right\rangle \\
&= \left\langle \delta_{\frac{1}{\sqrt{t_k}}} E(t_k)(E(-t_k)x_k - y), \delta_{\frac{1}{\sqrt{t_k}}} E(t_k)(E(-t_k)x_k - y) \right\rangle \\
&= \left\langle E(1)\delta_{\frac{1}{\sqrt{t_k}}}(E(-t_k)x_k - y), E(1)\delta_{\frac{1}{\sqrt{t_k}}}(E(-t_k)x_k - y) \right\rangle \\
&= \left\langle E(1)^T E(1)\delta_{\frac{1}{\sqrt{t_k}}}(E(-t_k)x_k - y), \delta_{\frac{1}{\sqrt{t_k}}}(E(-t_k)x_k - y) \right\rangle \\
&\geq \lambda_2 \left| \delta_{\frac{1}{\sqrt{t_k}}}(E(-t_k)x_k - y) \right|^2 \geq \lambda_2 t_k^{-Q} \left|(E(-t_k)x_k - y)\right|^2
\end{aligned}$$

for some constant $\lambda_2 > 0$, since $E(1)^T E(1)$ is positive definite and $t_k \in (0, 1)$.

The latter two estimates combined show

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus B_\delta(x)} \Gamma(t_k, x_k, 0, y) \mathrm{d}y &\leq \frac{c_0}{t_k^{\frac{Q}{2}}} \int_{\mathbb{R}^N \setminus B_\delta(x_0)} \exp\left(-\frac{\lambda_1 \lambda_2}{36 t_k^Q} |y - x|^2 \right) \mathrm{d}y \\
&\leq c_1 \int_\delta^\infty t_k^{-\frac{Q}{2}} \exp\left(-\frac{c_2}{t_k^Q} r^2 \right) r^{N-1} \mathrm{d}r \qquad (3.1.10)
\end{aligned}$$

for some constants $c_1, c_2 > 0$ and all $t_k \in (0, t_0)$. The integrand in equation (3.1.10) converges pointwise to 0 as $k \to \infty$. Moreover, since $(\delta, \infty) \to \mathbb{R}$, $r \mapsto t_k^{-\frac{Q}{2}} \exp(-c_2 t_k^{-Q} \frac{r^2}{2})$ is bounded independently of $k \in \mathbb{N}$ the function $r \mapsto c_3 \exp(-c_2 \frac{r^2}{2}) r^{N-1}$ is an integrable majorant for some constant $c_2 > 0$. An application of the theorem of dominated convergence shows that

$$\lim_{k \to \infty} \int_{\mathbb{R}^N \setminus B_\delta(x)} \Gamma(t_k, x_k, 0, y) \mathrm{d}y = 0$$

as $k \to \infty$. □

Lemma 3.1.17. The fundamental solution corresponding to $t_1 = 0$ is a Schwartz function, i.e. $\Gamma(t, \cdot, 0, y) \in \mathcal{S}$ for all $t > 0$ and any $y \in \mathbb{R}^N$. Moreover, it holds $\Gamma(\cdot, \cdot, 0, y) \in C^\infty((0, \infty) \times \mathbb{R}^N; \mathbb{R}^N)$.

Proof. For every $t > 0$ and any $y \in \mathbb{R}^N$ the function $x \mapsto \Gamma(t, x, 0, y)$ is infinitely often differentiable. Moreover, it can be seen by an induction argument that every derivative is given by a polynomial multiplied by the exponential function of a polynomial in x. This is the reason which implies $\Gamma(t, \cdot, 0, y) \in \mathcal{S}$. To obtain the differentiability for positive times, we note that by lemma 3.1.9, the term appearing in the exponential is a rational function in t with pole 0. $\square$

Lemma 3.1.18. Let $T > 0$, $0 < t_1 < t_2 < T$ and $x_1, x_2 \in \mathbb{R}^N$, then

$$[\nabla_{x_2}\Gamma](t_2, x_2, t_1, x_1) = -\frac{1}{2}\mathcal{C}^{-1}(t_2 - t_1)(x_2 - E(t_2 - t_1)x_1)\Gamma(t_2, x_2, t_1, x_1).$$

Moreover, there is a constant $c_1 = c_1(T) > 0$ such that

$$|[\nabla_{x_2}\Gamma](t_2, x_2, t_1, x_1)| \leq \frac{c_1}{(t_2 - t_1)^{2r+1}}\Gamma(t_2, x_2, t_1, x_1)\,|x_2 - E(t_2 - t_1)x_1|$$

for all $0 < t_1 < t_2 < T$ and any $x_1, x_2 \in \mathbb{R}^N$.

Proof. The formula for the gradient of the fundamental solution follows by chain rule, recalling that

$$\nabla \langle Sx, x\rangle = 2Sx$$

for any symmetric matrix $S \in \mathbb{R}^{N\times N}$. Let $0 < t_1 < t_2 < T$, then $t_2 - t_1 < T$ and therefore $(t_2 - t_1)^{-2k-1} \leq \tilde{c}_1(t_2 - t_1)^{-2r-1}$ for all $k = 0, \ldots, r$ and some constant $\tilde{c}_1 = \tilde{c}_1(T) > 0$. We deduce

$$\begin{aligned}
|[\nabla\Gamma](t_2, x_2, t_1, x_1)| &\leq \frac{1}{2}\left|\mathcal{C}^{-1}(t_2 - t_1)(x_2 - E(t_2 - t_1)x_1)\right| |\Gamma(t_2, x_2, t_1, x_1)| \\
&= \frac{1}{2}\left\|\delta_{\frac{1}{\sqrt{t_2 - t_1}}}\mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t_2 - t_1}}}\right\| \Gamma(t_2, x_2, t_1, x_1)\,|x_2 - E(t_2 - t_1)x_1| \\
&\leq \frac{\tilde{c}_1\,\|\mathcal{C}^{-1}(1)\|}{2(t_2 - t_1)^{2r+1}}\Gamma(t_2, x_2, t_1, x_1)\,|x_2 - E(t_2 - t_1)x_1|\,.
\end{aligned}$$

$\square$

We introduce the formal adjoint of $\mathcal{K} - \partial_t$, given by

$$(\mathcal{K} - \partial_t)^T = \mathcal{K}^T + \partial_t = \mathrm{div}(A\nabla) - \langle x, B\nabla\rangle + \partial_t.$$

Proposition 3.1.19. Let $t_0 > 0$ and $x_0 \in \mathbb{R}^N$, then

$$[(\mathcal{K} - \partial_t)^T\Gamma(t_0, x_0, \cdot, \cdot)](t, x) = 0$$

for all $t < t_0$ and any $x \in \mathbb{R}^N$.

Proof. We are going to show the claim first in the case that $t_0 = 0$ and $x_0 = 0$. The equation for the formal adjoint of $(\mathcal{K} - \partial_T)^T$ is equivalent to the equation

$$\partial_t u = -\operatorname{div}(A\nabla u) + \langle x, B\nabla u\rangle.$$

Using the substitution $v(t,x) = u(-t,x)$, we transform the latter equation to

$$\partial_t v = \operatorname{div}(A\nabla v) + \langle x, (-B)\nabla v\rangle.$$

We have already calculated that the fundamental solution with respect to the coefficients A and $-B$ solves this equation. Keeping in mind the substitution $u(t,x) = v(-t,x)$, we deduce that the function u defined as

$$u(t,x) = \left((4\pi)^N \det \tilde{\mathcal{C}}(-t)\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{4}\left\langle \tilde{\mathcal{C}}^{-1}(-t)x, x\right\rangle\right)$$

for $t < 0$ and $x \in \mathbb{R}^N$, solves the adjoint equation $(\mathcal{K} - \partial_t)^T = 0$. Here $\tilde{\mathcal{C}}$ denotes the matrix-valued function $\mathcal{C}$ from definition 3.1.4 with respect to the matrices A and $-B$. Using an argument similar to equation (3.1.4), we deduce

$$\begin{aligned} u(t,x) &= \left((4\pi)^N \det \tilde{\mathcal{C}}(-t)\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{4}\left\langle \mathcal{C}^{-1}(-t)(E(-t)x), E(-t)x\right\rangle\right) \\ &= \Gamma(0,0,t,x), \end{aligned}$$

where $\mathcal{C}$ and E denote the matrix-valued functions with respect to A and B. The general case follows by an argument similar to that presented in remark 3.1.7. $\square$

3.1.2 The classical Cauchy problem

Theorem 3.1.20. *For every $f \in C_b(\mathbb{R}^N)$ and any $T > 0$ there exists a unique classical solution $u \in C([0,\infty) \times \mathbb{R}^N) \cap C^{1,2}((0,\infty) \times \mathbb{R}^N)$ which is bounded on $[0,T] \times \mathbb{R}^N$ solving the Cauchy problem*

$$\begin{cases} \partial_t u(t,x) = \operatorname{div}\left(A\nabla u(t,x)\right) + \langle x, B\nabla u(t,x)\rangle, & t > 0, x \in \mathbb{R}^n \\ u(0,x) = f(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.1.11}$$

Furthermore, it holds

$$\sup_{t\geq 0} \|u(t,\cdot)\|_{\infty,\mathbb{R}^N} \leq \|f\|_{\infty,\mathbb{R}^N}.$$

The function u can be written as

$$u(t,x) = \int_{\mathbb{R}^N} \Gamma(t,x,0,y) f(y) \mathrm{d}y \tag{3.1.12}$$

for every $t > 0$ and $x \in \mathbb{R}^N$. This defines a semigroup $(T(t))_{t\geq 0}$ on $C_b(\mathbb{R}^n)$ given by $[T(t)f](x) = u(t,x)$ for any $f \in C_b(\mathbb{R}^N)$.

Proof. Let us start with the existence of classical solutions. Let u be defined as in (3.1.12) for $t > 0$ and $u(0,x) = f(x)$. Then by lemma 3.1.17 and interchanging differentiation and integration, it holds $u \in C((0,\infty)\times\mathbb{R}^N) \cap C^{1,2}((0,\infty)\times\mathbb{R}^N)$. Recalling remark 3.1.7, we deduce that u solves the partial differential equation in $(0,\infty)\times\mathbb{R}^N$. Further, by lemma 3.1.14, we have

$$\sup_{t\geq 0} \|u(t,\cdot)\|_{\infty,\mathbb{R}^N} \leq \int_{\mathbb{R}^N} \Gamma(t,x,0,y)\mathrm{d}y \|f\|_{\infty,\mathbb{R}^n} = \|f\|_{\infty,\mathbb{R}^n}.$$

We show that u continuously attains its initial value. Let $x \in \mathbb{R}^N$, $t_k \to 0$ and $x_k \to x$, then by theorem 3.1.16, the sequence $(\Gamma(t_k,\cdot,0,\cdot))_{k\in\mathbb{N}}$ defines a generalized Dirac sequence. By proposition A.3.3, we conclude that $u(t_k,x_k) \to f(x)$. This shows that $u \in C([0,\infty)\times\mathbb{R}^N) \cap C^{1,2}((0,\infty)\times\mathbb{R}^N)$. It remains to show that bounded classical solutions are unique. Let $T > 0$. By linearity, it suffices to show that if $f = 0$, the solution is equal to 0 for all $t \in [0,T]$. We are going to show that $u \leq 0$ in $[0,T]\times\mathbb{R}^n$. The same argument can be repeated with $-u$ to deduce $u \geq 0$ and thus $u = 0$. We want to apply the maximum principle from proposition 2.1.1. To do so, we choose the coercive and nonnegative function $\varphi(x) = |x|^2$ and $\lambda_0 = 2\|B\|$. It holds

$$K\varphi = \mathrm{tr}(A) + 2\langle x, Bx\rangle \leq \mathrm{tr}(A) + \lambda_0 |x|^2$$

and since u is bounded, we also know that

$$\limsup_{|x|\to\infty} \sup_{t\in[0,T]} \frac{u(t,x)}{\varphi(x)} = 0.$$

Proposition 2.1.1 implies $u \leq 0$. This shows the uniqueness of bounded solutions. □

Remark 3.1.21. By making use of Duhamel's principle, one can also show existence of solutions to the non-homogenous problem. We omit an implementation of this technique and refer to [Eva10, Chapter 2, Theorem 2] for a presentation of an analog situation, the heat equation. These calculations can be transferred to the Kolmogorov equation.

We are going to show that uniqueness of solutions for (3.1.11) holds also in a wider class of functions than in the class of bounded classical solutions. This statement is proven in [Pol95] for variable coefficients. For the sake of simplicity we mimic these arguments in the case of constants coefficients.

Theorem 3.1.22. *Let* $u \in C([0,T] \times \mathbb{R}^N) \cap C^{1,2}((0,T) \times \mathbb{R}^N)$ *be a classical solution of the Cauchy problem* (3.1.11) *for* $f = 0$*. If*

$$\int_0^T \int_{\mathbb{R}^N} \exp(-c\,|x|^2)\,|u(t,x)|\,\mathrm{d}x\mathrm{d}t < \infty$$

for some constant $c > 0$*, then* $u = 0$*.*

Proof. Using the product rule, we obtain the following Green-type identity

$$v(\mathcal{K} - \partial_t)u - u(\mathcal{K}^T + \partial_t)v = \operatorname{div}(vA\nabla u - uA\nabla v) + \langle x, B\nabla(uv)\rangle - \partial_t(uv). \tag{3.1.13}$$

Let $x_0 \in \mathbb{R}^N$. For $R > 0$ we introduce the cutoff function $\eta_R(x) = \eta(R^{-1}\,|x - x_0|)$, where η is the one dimensional cutoff function defined in section A.3. As seen in lemma A.3.6, it holds $\operatorname{supp}(\nabla\eta_R) \subset B_{2R}(x_0)\backslash B_R(x_0)$ and $|\nabla\eta_R| \le c_1 R^{-1}$ for some constant $c_1 > 0$. Consequently, as we have often used in section 2.2, it is $|Y\eta_R| \le c_2$ for some constant $c_2 > 0$. Moreover, if $R \ge 1$, it is $\left|\mathcal{K}^T\eta_R\right| \le c_3$ for some constant $c_3 > 0$ independent of $R \ge 1$. Let $t_0 \in (0,T)$ and $\delta \in (0,t_0)$, then, by integrating the Green-type identity (3.1.13) for the solution u and $v(s,\xi) = \eta_R(\xi)\Gamma(t_0,x_0,s,\xi)$ over $(0,t_0-\delta) \times B_{2R}(x_0)$, we obtain

$$\begin{aligned}
-\int_0^{t_0-\delta}\int_{B_{2R}(x_0)} u(\mathcal{K}-\partial_t)^T v\mathrm{d}\xi\mathrm{d}s &= \int_0^{t_0-\delta}\int_{B_{2R}(x_0)} \operatorname{div}(vA\nabla u - uA\nabla v) + \langle \xi, B\nabla(uv)\rangle - \partial_t(uv)\mathrm{d}\xi\mathrm{d}s \\
&= -\int_{B_{2R}(x_0)} u(\xi, t_0-\delta)\eta_R(\xi)\Gamma(t_0,x_0,t_0-\delta,\xi)\mathrm{d}\xi \\
&\quad + \int_{B_{2R}(x_0)} u(0,\cdot)\eta_R\Gamma(t_0,x_0,\cdot)\mathrm{d}\xi \\
&\quad + \int_0^{t_0-\delta}\int_{\partial B_{2R}(x_0)} \langle vA\nabla u - uA\nabla v, \xi\rangle\,\mathrm{d}\mathcal{S}(\xi)\mathrm{d}s \\
&\quad - \int_0^{t_0-\delta}\int_{\partial B_{2R}(x_0)} \langle \xi, B\xi\rangle\, uv\mathrm{d}\mathcal{S}(\xi)\mathrm{d}s,
\end{aligned}$$

since $(\mathcal{K} - \partial_t)u = 0$ and by using the divergence theorem. Furthermore, the last two terms are zero, since $\eta_R = 0$ on $\partial B_{2R}(x_0)$. The term third from last is zero, due to the assumption $u(0) = 0$. Let $\varepsilon > 0$, then, due to the continuity of u, there is a $\delta' > 0$ small such that $|u(\xi,t_0-\delta)\eta_R(\xi) - u(x_0,t_0)| < \varepsilon$ for all $\delta < \delta'$ and all $\xi \in \mathbb{R}^N$ such that $|x_0 - \xi| < \delta'$. The equation $\Gamma(t_0,x_0,t_0-\delta,\xi) = \Gamma(\delta,x_0,0,\xi)$ shows that we can use proposition A.3.3 for the bounded sequence of functions $u(\xi,t_0-\delta)\eta_R(\xi)$, the constant sequence x_0 and the generalized Dirac sequence $(\Gamma(\delta,\cdot,0,\cdot))_{\delta>0}$ to deduce

$$u(t_0,x_0)=\int_0^{t_0}\int_{B_{2R}(x_0)}u(s,\xi)(\mathcal{K}-\partial_t)^T v(s,\xi)\mathrm{d}\xi\mathrm{d}s$$

by taking the limit $\delta\to 0^+$ in the latter equation. Proposition 3.1.19 implies

$$(\mathcal{K}-\partial_t)^T\Gamma(t_0,x_0,\cdot,\cdot)=0$$

and thus

$$\begin{aligned}u(t_0,x_0)&=\int_0^{t_0}\int_{B_{2R}}u\eta_R(\mathcal{K}-\partial_t)^T\Gamma+u\Gamma(\mathcal{K}-\partial_t)^T\eta_R+2u\left\langle A\nabla\eta_R,\nabla_\xi\Gamma(t_0,x_0,s,\xi)\right\rangle\mathrm{d}\xi\mathrm{d}s\\&=\int_0^{t_0}\int_{B_{2R}}u\Gamma\mathcal{K}^T\eta_R+2u\left\langle A\nabla\eta_R,\nabla_\xi\Gamma(t_0,x_0,s,\xi)\right\rangle\mathrm{d}\xi\mathrm{d}s,\end{aligned}$$

since $\nabla\eta_R=0$ on $B_R(x_0)$ and $\operatorname{supp}\eta_R\subset B_{2R}(x_0)$. We estimate by lemma B.0.6

$$\begin{aligned}|u(t_0,x_0)|&\le\int_0^{t_0}\int_{B_{2R}(x_0)}\left(\Gamma\left|\mathcal{K}^T\eta_R\right|+2\left|\langle A\nabla\eta_R,\nabla_\xi\Gamma(t_0,x_0,s,\xi)|\rangle\right)|u|\,\mathrm{d}\xi\mathrm{d}s\\&\le\int_0^{t_0}\int_{B_{2R}(x_0)}\left(c_3\Gamma+2\left\|A\right\|\left|\nabla\eta_R\right|\left|\nabla_\xi\Gamma(t_0,x_0,s,\xi)\right|\right)|u|\,\mathrm{d}\xi\mathrm{d}s\\&\le c_4\int_0^{t_0}\int_{B_{2R}(x_0)}\left(1+\frac{1}{R}\frac{1}{(t_0-s)^{2r+1}}\left|x_0-E(t_0-s)\xi\right|\right)\Gamma\,|u|\,\mathrm{d}\xi\mathrm{d}s\\&\le c_5\int_0^{t_0}\int_{B_{2R}(x_0)}\left(1+\frac{1}{(t_0-s)^{2r+1}}|x_0|+\frac{1}{(t_0-s)^{2r+1}}\left\|E(t_0-s)\right\||\xi|\right)\Gamma\,|u|\,\mathrm{d}\xi\mathrm{d}s\\&\le c_6\int_0^{t_0}\int_{B_R(x_0)^c}\left(1+\frac{1}{(t_0-s)^{2r+1}}+\frac{|\xi|}{(t_0-s)^{2r+1}}\right)\Gamma(t_0,x_0,s,\xi)\,|u(s,\xi)|\,\mathrm{d}\xi\mathrm{d}s\end{aligned}$$

for constants $c_4,c_5,c_6>0$ independent of $R\ge R(x_0)$, since $\sup_{s\in[0,t_0]}\left\|E(t_0-s)\right\|<\infty$. Using the estimate from lemma B.0.6, we get

$$\begin{aligned}&|u(t_0,x_0)|\\&\le c_7\int_0^{t_0}\int_{B_R(x_0)^c}t^{-\frac{Q}{2}}\left(1+\frac{1}{(t_0-s)^{2r+1}}+\frac{|\xi|}{(t_0-s)^{2r+1}}\right)\exp\left(-\frac{c_8}{(t_0-s)}|\xi|^2\right)|u(s,\xi)|\,\mathrm{d}\xi\mathrm{d}s\end{aligned}$$

for constants $c_7,c_8>0$, a possible reduction of $t_0<t_1$ and $R\ge R(x_0)$. We choose

$\varepsilon = \min\left(1, \frac{c_8}{2c}, t_1\right) > 0$ and note that it holds

$$\sup_{s\in[0,t_0),\, \xi\in B_R^c(x_0)} \frac{1}{t^{\frac{Q}{2}}} \left(1 + \frac{1}{(t_0-s)^{2r+1}} + \frac{|\xi|}{(t_0-s)^{2r+1}}\right) \exp\left(-\frac{c_8}{2(t_0-s)}|\xi|^2\right) < \infty$$

independently of $R \geq R(x_0)$, whence

$$|u(t_0,x_0)| \leq c_8 \int_0^{t_0} \int_{B_R(x_0)^c} \exp\left(-\frac{c_8}{2(t_0-s)}|\xi|^2\right) |u(s,\xi)| \,\mathrm{d}\xi\mathrm{d}s$$

for some constant $c_8 > 0$ independent of $R \geq R(x_0)$. Consequently, due to the choice of $t_0 < \varepsilon$, we deduce

$$|u(t_0,x_0)| \leq c_8 \int_0^{t_0} \int_{B_R(x_0)^c} \exp\left(-c|\xi|^2\right) |u(s,\xi)| \,\mathrm{d}\xi\mathrm{d}s$$

for all $R \geq R(x_0)$, where c is the constant given in the assumption. By assumption, the integral on the right-hand side is finite for $R = 0$. This shows that the right-hand side converges to 0 as $R \to \infty$. Consequently, $u(t_0,x_0) = 0$ and hence $u(t,x) = 0$ for all $(t,x) \in [0,\varepsilon)\times\mathbb{R}^N$. To conclude, we divide $[0,T]$ into finitely many equal parts of length smaller than ε. Applying the proven result on each of these intervals successively shows the claim. □

We highlight the fact that the uniqueness classes are the same uniqueness classes that are known for the heat equation. Using the last theorem, one can show the following uniqueness theorem by proving a representation inequality for nonnegative solutions in terms of the fundamental solution.

Theorem 3.1.23. *Every nonnegative solution* $0 \leq u \in C([0,T]\times\mathbb{R}^N) \cap C^{1,2}((0,T)\times\mathbb{R}^N)$ *of the Cauchy problem* (3.1.11) *with* $f = 0$ *is identical zero in* $[0,T]\times\mathbb{R}^N$.

Proof. [Pol95, Theorem 3.2] □

A consequence of the latter theorem is the following representation theorem for nonnegative classical solutions.

Theorem 3.1.24. *Let* $u \in C([0,T]\times\mathbb{R}^N) \cap C^{1,2}((0,T)\times\mathbb{R}^N)$ *be a nonnegative solution of the Cauchy problem* (3.1.11)*, then*

$$u(t,x) = \int_{\mathbb{R}^N} \Gamma(t,x,s,y) u(s,y) \mathrm{d}y$$

holds for all $t > s \geq 0$ *and* $x \in \mathbb{R}^N$.

Proof. [Pol95, Corollary 3.1, Proposition 3.2] □

3.1.3 The Cauchy problem with initial data in $L^p(\mathbb{R}^N)$

Theorem 3.1.25. *For all $p \in [1, \infty)$ the restriction of the semigroup defined by* (3.1.12) *to $C_c^\infty(\mathbb{R}^N)$ can be extended to a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $L^p(\mathbb{R}^N)$. Moreover, for every $f \in L^p(\mathbb{R}^N)$ it holds*

$$[T(t)f](x) = \int_{\mathbb{R}^N} \Gamma(t, x, 0, y) f(y) \mathrm{d}y \tag{3.1.14}$$

for all $t > 0$ and every $x \in \mathbb{R}^N$. Further, $(T(t))_{t\geq 0}$ is a contractive semigroup.

Proof. We want to write the semigroup $T(t)$ as the composition of two operators. Let us introduce the group $S(t)$ on $L^p(\mathbb{R}^N)$ defined by $[S(t)f](x) = f(\exp(tB^T)x)$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. For $t > 0$ and $f \in C_c^\infty(\mathbb{R}^N)$ we further define

$$[G_t f](x) = \frac{c_0}{t^{\frac{Q}{2}}} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{4}\left\langle \mathcal{C}(t)^{-1}(x-y), x-y\right\rangle\right) f(y)\mathrm{d}y = (\Gamma(t, \cdot) * f)(x). \tag{3.1.15}$$

So for every $f \in C_c^\infty(\mathbb{R}^N)$ we have $T(t)f = G_t \circ S(t)f$.

Let us examine the group $S(t)$ first. If $f \in C_c^\infty(\mathbb{R}^N)$, it holds $\|S(t)f - f\|_{\infty,\mathbb{R}^N} \to 0$ as $t \to 0$. Let $|t| < 1$, then there is a ball $B \subset \mathbb{R}^N$ such that $\operatorname{supp} S(t)f \subset B$ for all $|t| < 1$. This shows the convergence $\|S(t)f - f\|_{p,\mathbb{R}^N} \to 0$ for $t \to 0$. It holds

$$\|S(t)f\|_{p,\mathbb{R}^N}^p = \int_{\mathbb{R}^N} \left|f(\exp(tB^T)x)\right|^p \mathrm{d}x = \int_{\mathbb{R}^N} |f(y)|^p \mathrm{d}y = \|f\|_{p,\mathbb{R}^N}^p, \tag{3.1.16}$$

since $\det(\exp(-tB^T)) = 1$ for all $t \geq 0$. By the equicontinuity lemma B.0.7, we deduce that $\lim\limits_{t\to 0} S(t)f = f$ for all $f \in L^p(\mathbb{R}^N)$.

Let $t > 0$ and $f \in L^p(\mathbb{R}^N)$, then

$$\|G_t f\|_{p,\mathbb{R}^N} = \|\Gamma(t, \cdot) * f\|_{p,\mathbb{R}^N} \leq \|\Gamma(t, \cdot)\|_{1,\mathbb{R}^N} \|f\|_{p,\mathbb{R}^N} = \|f\|_{p,\mathbb{R}^N}$$

by Young's inequality together with lemma 3.1.14. To conclude, we note that by theorem 3.1.16 it holds that $(\Gamma(t, \cdot))_{t>0}$ is a Dirac sequence and therefore $\|G_t f - f\|_{p,\mathbb{R}^N} = \|\Gamma(t, \cdot) * f - f\|_{p,\mathbb{R}^N} \to 0$ as $t \to 0$ by proposition A.3.4. Using the equicontinuity lemma B.0.7, we deduce $\lim_{t\to 0} T(t)f = f$ in $L^p(\mathbb{R}^N)$ for any $f \in L^p(\mathbb{R}^N)$. Since $C_c^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$, the semigroup is indeed the unique extension of the semigroup defined in equation (3.1.12) restricted to $C_c^\infty(\mathbb{R}^N)$. □

Proposition 3.1.26. Let us denote by $T(t)$ the semigroup defined in equation (3.1.12) on $L^p(\mathbb{R}^N)$ or $C_b(\mathbb{R}^N)$. It holds

(i) $T(t)\mathbb{1} = \mathbb{1}$,

(ii) $T(t)\mathcal{S}(\mathbb{R}^N) \subset \mathcal{S}(\mathbb{R}^N)$,

(iii) $T(t) \geq 0$

for all $t \geq 0$.

Proof. Property (i) is an immediate consequence of lemma 3.1.14. The second property can be seen by taking the Fourier transform in the space variable and noting that $\mathcal{S}$ is invariant under Fourier transform. The third property is a direct consequence of the fact that $\Gamma(t, x, 0, y)$ is positive.

□

Remark 3.1.27. It remains to investigate the generator $(\mathcal{K}_p, D(\mathcal{K}_p))$ of $T(t)$ in $L^p(\mathbb{R}^N)$. In the nondegenerate case, i.e. if A is symmetric positive definite, one can show that the generator is given by the differential operator $\mathcal{K}_p u = \operatorname{div}(A\nabla u) + \langle u, B\nabla u\rangle$ with domain $D(\mathcal{K}_p) = \{u \in W^{2,p}(\mathbb{R}^N) \mid \mathcal{K}_p u \in L^p(\mathbb{R}^n)\}$ for $p \in (1, \infty)$. A proof of this statement can be found in [Lor17, Section 10.4] and in [Met01]. The crucial arguments therein are based on the ellipticity of the diffusion matrix A. Using the results from section 2.2, the next theorem gives a first step towards a characterization of the generator in the degenerate case.

Theorem 3.1.28. *For $p \in (1, \infty)$ the domain of the generator $\mathcal{K}_p$ is given by*

$$D(\mathcal{K}_p) = \{u \in L^p(\mathbb{R}^N) \mid \mathcal{K}_p u \in L^p(\mathbb{R}^N)\}$$

where $\mathcal{K}_p u$ is interpreted in the distributional sense. In particular, $(C_c^\infty(\mathbb{R}^N), \mathcal{K}_p)$ is a core of the generator.

Proof. One can show that the Schwartz functions $\mathcal{S}$ are a core of $(\mathcal{K}_p, D(\mathcal{K}_p))$. The idea for this proof can be found in [Met01, Proposition 3.2]. More rigorous arguments in a similar situation are provided in [Ott17, Theorem 3.2]. We note that in both cases the statement is proven in the nondegenerate case. However, in each proof only the definiteness of $\mathcal{C}(t)$ is of importance. So that one can transfer these arguments to the degenerate case. The theorem is then the consequence of theorem 2.2.16 and lemma A.1.2.

□

3.2 Kolmogorov equations with variable diffusion coefficients

3.2.1 A semigroup approach

In this section we are going to present a result on well-posedness of a class of Kolmogorov equations with variable coefficients based on the methods presented in section 2.2. We consider the differential operator

$$\mathcal{K}u = \operatorname{div}(A(x)\nabla u) + \langle x, B\nabla u\rangle \tag{3.2.1}$$

where $B \in \mathbb{R}^{N\times N}$ is a constant matrix and $A \in C^2(\mathbb{R}^N; \mathbb{R}^{N\times N})$ with bounded second derivatives such that $A(x)$ is symmetric positive semidefinite for all $x \in \mathbb{R}^N$. Let $p \in (1, \infty)$, then the maximal realization is the operator $\mathcal{K}_p u = \mathcal{K}u$ with domain $D(\mathcal{K}_p) = \{u \in L^p(\mathbb{R}^n) \mid \mathcal{K}u \in L^p(\mathbb{R}^n)\}$. The results obtained in section 2.2 lead to the following theorem.

Theorem 3.2.1. *The operator $(\mathcal{K}_p, D(\mathcal{K}_p))$ is the generator of a quasicontractive and positive C_0-semigroup $T(t)$. In particular, there is an $\omega \in \mathbb{R}$ satisfying*

$$\|T(t)f\|_{p,\mathbb{R}^N} \leq \exp(\omega t)\|f\|_{p,\mathbb{R}^N}$$

for all $t \geq 0$ and all $f \in L^p(\mathbb{R}^N)$. $(\mathcal{K}, C_c^\infty(\mathbb{R}^N))$ is a core of the generator. Moreover, for every $f \in L^p(\mathbb{R}^N)$ there exists a unique weak solution $u \in C([0,\infty), L^p(\mathbb{R}^N))$ of the Cauchy problem

$$\begin{cases} \partial_t u = \mathcal{K}_p u, & t \geq 0 \\ u(0) = f. \end{cases} \tag{3.2.2}$$

If additionally $f \in D(\mathcal{K}_p)$, then u is a strong solution. The semigroup is positive so that if $f \geq 0$, it holds $u(t) \geq 0$ for all $t \geq 0$. Finally, the space $W^{1,p}(\mathbb{R}^n)$ is an invariant set of $T(t)$ and there is some nonnegative constant γ such that it holds $\|T(t)f\|_{1,p,\mathbb{R}^N} \leq \exp(\gamma t)\|f\|_{1,p,\mathbb{R}^N}$ for every $f \in W^{1,p}(\mathbb{R}^N)$ and any $t \geq 0$.

Proof. It suffices to show that the coefficients satisfy the assumptions (A1) and (A2). Assumption (A2) and the condition on the diffusion coefficients are trivial. Note that our drift term $\langle x, B\nabla\rangle$ only grows linear in x so that it is admissible, too. □

Remark 3.2.2. We want to comment that it is allowed that the diffusion matrix may vanish and thus $\mathcal{K}$ can also be the first order kinetic transport operator. The reader, who is already familiar with the concept of hypoellipticity, in particular the results from section 6.1, may wonder why this assumption, i.e. the structural properties on the diffusion matrix and the drift matrix were not assumed in this section. One might be tempted to think that under the hypoellipticity assumption one could remove the part in section 2.2, where one performs the elliptic regularization to simplify the proof of the generator property. While hypoellipticity provides us with smooth solutions and local estimates on the L^p norm, it does not provide global bounds as for example obtained in proposition 2.2.8. We refer to [RS76] for more information on these local L^p estimates. Furthermore, to use the theory of hypoelliptic operators one would need to assume that the coefficients were in C^∞.

3.2.2 Irregular diffusion coefficients

In this section we are going to review a result from [BL08]. Therein the authors prove existence and uniqueness of weak solutions to Kolmogorov equations with variable diffusion coefficients under the assumption of linear growth and Lipschitz continuity. Let $n \in \mathbb{N}$ and $N = 2n$. We consider the Cauchy problem

$$\begin{cases} \partial_t u(t,v,x) + v \cdot \nabla_x u(t,v,x) = \operatorname{div}(A^T(v,x)A(v,x)\nabla u(t,v,x)) & t > 0, (v,x) \in \mathbb{R}^{2n} \\ u(0,v,x) = f(v,x) & (v,x) \in \mathbb{R}^{2n} \end{cases} \tag{3.2.3}$$

where $A\colon \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is a measurable map and $f \in L^2 \cap L^\infty(\mathbb{R}^N)$. Given $T > 0$, a weak solution is to be understood as a function $p \in L^\infty([0,T]; L^2 \cap L^\infty(\mathbb{R}^N))$ satisfying $A\nabla u \in (L^2([0,T]; L^2(\mathbb{R}^N)))^N$ such that

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}^N} u\partial_t\varphi \mathrm{d}(v,x)\mathrm{d}t - \int_{\mathbb{R}^N} f\varphi(0,\cdot)\mathrm{d}(v,x) - \int_0^T \int_{\mathbb{R}^N} u(t,v,x) \langle v, \nabla_x\varphi\rangle \mathrm{d}(v,x)\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^N} \langle A\nabla u, A\nabla\varphi\rangle \mathrm{d}(v,x)\mathrm{d}t \end{aligned} \tag{3.2.4}$$

for all $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^N)$. The authors of [BL08] prove the following theorem.

Theorem 3.2.3. *Let $m \in \mathbb{N}$ and $A \in \left(W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N)\right)^{N\times m}$ such that*

$$(1 + |(v,x)|)^{-1} A \in \left(L^\infty(\mathbb{R}^N)\right)^{N\times m}.$$

In this case there exists a unique weak solution of equation (3.2.3)*.*

Proof. This is a special case of [BL08, Proposition 1]. Note that we choose $b(v,x) = (0,v)^T$ so that $\operatorname{div} b = 0$ and clearly $(1 + |(v,x)|)^{-1} b \in L^\infty(\mathbb{R}^N)^N$. □

3.3 Kolmogorov equations on (bounded) domains

3.3.1 Bounded domains in velocity and position

In this section we are going to investigate how the theory presented in section 2.3 can be applied to the equation

$$\partial_t u + v \cdot \nabla_x u = \Delta_v u + h \tag{3.3.1}$$

on a bounded domain $Q = (0,T) \times \Omega$ for a suitable inhomogeneity h. Here $\Omega \subset \mathbb{R}^N$ is an open and bounded set with piecewise smooth boundary. Let us calculate the sigma partition of the boundary. To do so, we denote by $n = (n_v, n_x)$ the outer unit normal on $\partial\Omega$. Furthermore, the Fichera function F is given by

$$F(v,x) = \langle v, n_x \rangle$$

so that

$$\Sigma_T^0 = \{(t,v,x) \in \partial_p Q \mid \|n_v\| = 0\}$$

and therefore

$$\Sigma_-^T = \{0\} \times \Omega \cup (0,T) \times \{(v,x) \in \partial\Omega \mid \|n_v\| = 0 \text{ and } \langle v, n_x(v,x)\rangle < 0\}$$

as well as

$$\Sigma_c^T = (0,T) \times \{(v,x) \in \partial\Omega \mid \|n_v(v,x)\| \neq 0\}.$$

To apply the theory of section 2.3, we need a strictly negative zeroth order. To fix this problem, we introduce the equation

$$\partial_t w + v \cdot \nabla_x w + \lambda w = \Delta_v w + h$$

instead. If w is a solution of the latter equation, then $u = \exp(-\lambda t) w$ solves the Kolmogorov equation, at least formally. One might say that this is merely a technical problem. We want to investigate two examples.

Example 3.3.1. (i) Let us consider the rectangular set $\Omega = (-1,1) \times (-1,1) \subset \mathbb{R}^2$. As noted before, to apply the theory of Fichera, it suffices that Ω has piecewise smooth boundary. The boundary of Ω can be decomposed as

$$\partial\Omega = \overline{(-1,1)(1,1)} \cup \overline{(1,1)(1,-1)} \cup \overline{(1,-1)(-1,-1)} \cup \overline{(-1,-1)(-1,1)} =: \bigcup_{i=1}^{4} D_i.$$

In the interior of each of these lines D_i the outer unit normal n_i is given as

$$n_i = S^i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for $i = 1, \dots, 4$, where $S \in \mathbb{R}^{2\times 2}$ denotes the matrix representation of the clockwise rotation by $\frac{\pi}{2}$. The Fichera function is given by $F = vn_x$, where n_x denotes the x component of the outer unit normal n. We recall that for the sigma partition of $\partial_p Q$ we are only interested in the interior points of the lines. Consequently, we calculate

$$\begin{aligned} \Sigma_-^T = \{0\} \times \Omega \cup (0,T) \times \{(v,x) \in \mathbb{R}^2 \mid x = 1, v \in (-1,0)\} \\ \cup\, (0,T) \times \{(v,x) \in \mathbb{R}^2 \mid x = -1, v \in (0,1)\} \end{aligned}$$

and

$$\Sigma_c^T = (0,T) \times \{(v,x) \in \partial\Omega \mid v \in \{-1,1\}, x \in (-1,1)\}.$$

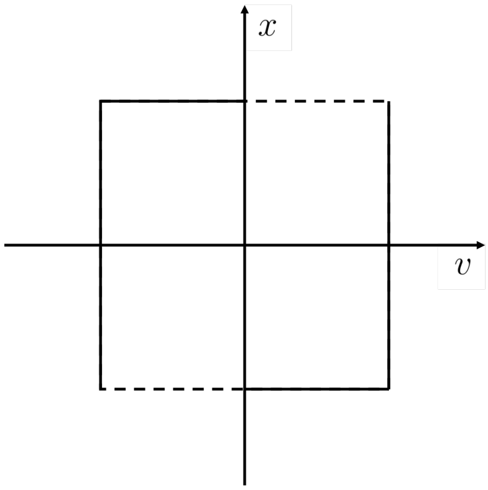

Figure 3.1: The partition of the boundary for the Kolmogorov equation on $(-1,1)^2$

The dotted lines in figure 3.1 are the part of the boundary where we are not allowed to prescribe boundary values.

(ii) Let us investigate the case $\Omega = B_1(0) \subset \mathbb{R}^2$. For every $(v,x) \in \partial B_1(0)$ the outer unit normal is given by $n = (v,x)$. Therefore, it holds

$$\Sigma^0 = \{(v,x) \in \partial B_1(0) \mid n_v = v = 0\},$$

whence

$$\Sigma_-^T = \{0\} \times \Omega$$

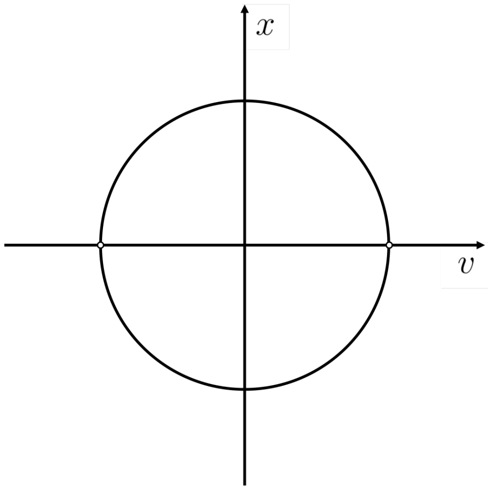

Figure 3.2: The partition of the boundary for the Kolmogorov equation on $\partial B_1(0)$

and

$$\Sigma_c^T = (0,T) \times \{(v,x) \in \partial B_1(0) \mid v \neq 0\} = (0,T) \times (\partial B_1(0) \setminus \{(0,1),(0,-1)\}).$$

The figure 3.4 shows the partition of the boundary in this situation. The only two points where we are not allowed to prescribe boundary values are marked by circles.

In both cases of the previous example we are able to apply theorem 2.3.2 to deduce the existence of weak solutions in the sense of definition 2.3.1 for zero initial value and homogenous boundary values on the set $(0,T) \times (\Sigma_- \cup \Sigma_c)$. Let us give a physical interpretation of the situation in the two mentioned examples. In the case of the rectangular domain the Kolmogorov equation describes the evolution of a particle on a one dimensional line with position $x \in (-1,1)$ and velocity $v \in (-1,1)$. Let us suppose that at a time t the particle is at the boundary part Σ_-, then due to the sign of velocity v, the particle is moving inwards. Thus, one can say that the boundary condition $u = 0$ on Σ_c^T describes that there is no influx of particles. In the second case we can keep the interpretation of a moving particle but now the attainable velocity of a particle at position x is bounded by $\pm\sqrt{1-x^2}$.

A more physical intuitive model would be to let the particle attain any velocity and only bound the domain of the position of the particle. In this case the domain of the partial differential equation is not bounded anymore and therefore we cannot apply the theory of Gaetano Fichera. We are going to see in the following section that one can prove existence of weak solutions under less restricted conditions on the boundary. Furthermore, we are also going to consider non-homogenous boundary conditions.

3.3.2 Particles in a bounded domain with arbitrary velocities

As announced in the previous section we are going to restrict only the domain of the space variable x and not the velocity variable v. From the physical point of view this seems very intuitive. Recalling the interpretation of moving particles, it makes sense that these particles

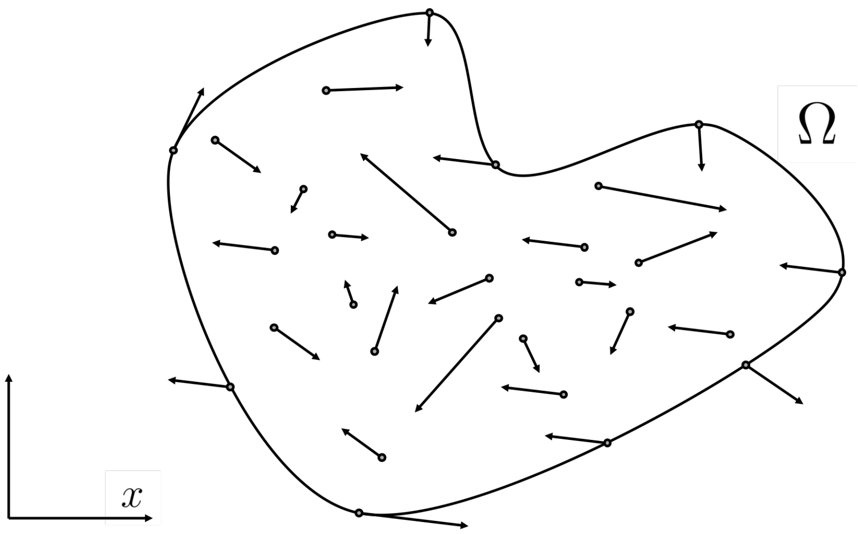

Figure 3.3: Movement of particles in a bounded domain

should be able to attain arbitrary velocities. The Kolmogorov equation then describes the evolution of multiple particles in a region given by a bounded set $\Omega \subset \mathbb{R}^n$ with velocities $v \in \mathbb{R}^N$. To be more precise, let $n \in \mathbb{N}$, $N = 2n$, $T > 0$, $\delta > 0$ and $\Omega \subset \mathbb{R}^N$ an open, bounded and smooth set. Let us denote by $n(x) \in \mathbb{R}^n$ the outer unit normal onto the boundary of Ω in $x \in \partial\Omega$. Given $x \in \partial\Omega$, we define the sets

$$\Sigma_{\pm}^{x} := \{v \in \mathbb{R}^n \mid \pm \langle v, n(x)\rangle > 0\}$$

and $\Sigma_{\pm} := \{(x, v) \in \partial\Omega \times \mathbb{R}^n \mid v \in \Sigma_{\pm}^{x}\}$. Further, we introduce the sets $Q_T = [0, T) \times \bar{\Omega} \times \mathbb{R}^n$, $\Sigma_{\pm}^{T} = (0, T) \times \Sigma_{\pm}$ and $\Sigma_T = (0, T) \times \partial\Omega \times \mathbb{R}^n$. Given a suitable function f, we are going to denote by $\gamma_{\pm} f$ the trace on $\Sigma_{\pm}^{T}$ and by γf the trace on Σ^T. We will come back to the traces on these sets at a later time. Using this notation, we are interested in the existence and uniqueness of appropriate weak solutions of

$$\begin{cases} \partial_t u + v \cdot \nabla_x u + \lambda u = \sigma \Delta_v u + h, & t > 0, x \in \Omega, v \in \mathbb{R}^n \\ u(t, v, x) = g(t, v, x), & (t, v, x) \in \Sigma_{-}^{T} \\ u(0, v, x) = f(v, x), & x \in \Omega, v \in \mathbb{R}^n \end{cases} \tag{3.3.2}$$

for appropriate functions f, g and constant $\lambda > 0$. It turns out that a suitable class of functions to describe the boundary function g is given by the sets

$$L^p(\Sigma^T_\pm) := \left\{ f\colon \Sigma^T_\pm \to \mathbb{R} \text{ measurable } \left| \int_{\Sigma^T_\pm} |f|^p \left|\langle v, n(x)\rangle\right| \mathrm{d}S(x)\mathrm{d}v\mathrm{d}t < \infty \right. \right\},$$

equipped with the norm $\|f\|_{p,\Sigma^T_\pm}$ with respect to above measure. Boundary conditions of the above type are called absorbing boundary conditions. Basically, they prescribe the inflow of particles at the boundary. This can be clarified by looking at figure 3.4. Let $x \in \partial\Omega$ and

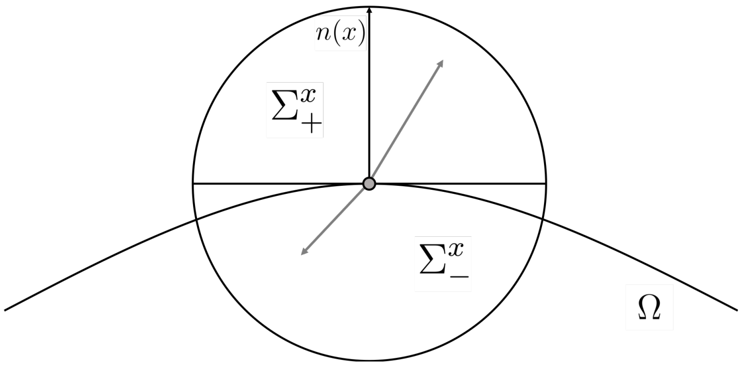

Figure 3.4: The partition of the boundary at $x \in \partial\Omega$.

$v \in \mathbb{R}^N$ such that $\langle v, n(x)\rangle < 0$. Such velocity vectors correspond to the lower semicircle in figure 3.4. They describe a particle at the boundary of Ω with inward pointing velocity vector. For example, if $g = 0$, there is no influx of particles. We note that the behavior of particles with outward pointing velocity, i.e. velocities in the upper semicircle in figure 3.4, is not fixed in the Cauchy problem (3.3.2). This is similar to the theory of Fichera where we weren't allowed to prescribe the outflow neither.

Another type of boundary conditions would be the so-called reflection-type boundary conditions where the quantities of inward and outward moving particles are set into relation by an equation of the form

$$\gamma_- f(t, v, x) = \int_{\Sigma^x_+} R(t, x; v, v')\gamma_+ f(t, v', x)\mathrm{d}v.$$

The kernel $R(t, x; v, v')$ is the probability that a particle striking the boundary in the point x at time t with velocity v is reflected with velocity v'. A special case is for example specular reflection which leads to

$$\gamma_- f(t, v, x) = \gamma_+ f(t, v - 2\langle v, n(x)\rangle n(x), x)\mathrm{d}v.$$

This section is based on the article [Car98] where the author uses the result to show existence of weak solutions to the initial and boundary value problem of the Vlasov-Poisson-Fokker-Planck system.

We are going to show existence of weak solutions in the Hilbert space setting. We denote by $Su = \partial_t u + v \cdot \nabla_x u$ the kinetic transport operator. Further, we introduce the Hilbert space

$$H = \{u \in L^2(Q_T) \mid \nabla_v u \in L^2(Q_T)\}$$

equipped with the scalar product $\langle f, g \rangle_H = \langle f, g \rangle_{L^2(Q_T)} + \langle \nabla_v f, \nabla_v g \rangle_{L^2(Q_T;\mathbb{R}^N)}$ and the subspace

$$\mathcal{T} = \{u \in H \,:\, \partial_t u + v \cdot \nabla_x u \in H'\}$$

with the prehilbertian scalar product $\langle f, g \rangle_{\mathcal{T}} = \langle f, g \rangle_H + \langle Sf, Sg \rangle_{H'}$ so that $S\colon \mathcal{T} \to H'$.

Definition 3.3.2. Let $T > 0$, $\lambda \geq 0$, $h \in L^2(Q_T)$, $g \in L^2(\Sigma_-^T)$ and $f \in L^2(\Omega \times \mathbb{R}^n)$. We call a function $u \in \mathcal{T}$ a weak solution of (3.3.2) if for all $\varphi \in C_c^\infty(Q_T)$ with $\varphi = 0$ on Σ_+^T, then it holds that

$$\begin{aligned}
&\int_{Q_T} -u\partial_t\varphi - uv \cdot \nabla_x\varphi + \sigma\nabla_v u\nabla_v\varphi \mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&= \int_{Q_T} h\varphi\mathrm{d}v\mathrm{d}x\mathrm{d}t + \int_{\Omega\times\mathbb{R}^n} f\varphi(0,\cdot)\mathrm{d}v\mathrm{d}x + \int_{\Sigma_-^T} g\varphi \left|\langle v, n(x)\rangle\right| \mathrm{d}S(x)\mathrm{d}v\mathrm{d}t.
\end{aligned}$$

Remark 3.3.3. Note that since $Q_T = [0, T) \times \bar{\Omega} \times \mathbb{R}^n$, the functions $\varphi \in C_c^\infty(Q_T)$ can also attain values for $t = 0$ and $x \in \partial\Omega$ so that the assumption $\varphi = 0$ on Σ_+^T is nontrivial.

The following theorem, which is due to Jacques-Louis Lions, will help us to show the existence of weak solutions. This theorem can be seen as a generalization of the classical Lax-Milgram theorem. The presented version of this theorem and its proof are taken from [Lio61, Chapter III, Theorem 1.1].

Theorem 3.3.4. *Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space with induced norm $\|\cdot\|_H$. Consider a subspace $F \subset H$ equipped with a prehilbertian scalarproduct $\langle\cdot,\cdot\rangle_F$ such that the injection of $(F, \|\cdot\|_F)$ into $(H, \|\cdot\|_H)$ is continuous. Given a bilinear form $a\colon H \times F \to \mathbb{R}$ such that*

(i) for all $\varphi \in F$ the map $a(\cdot, \varphi)$ is continuous on H,

(ii) there exists a constant $\alpha > 0$ such that $a(\varphi, \varphi) \geq \alpha \|\varphi\|_F^2$ for all $\varphi \in F$.

Then for every $L \in F'$ there exists a vector $u \in H$ such that

$$a(u, \varphi) = L(\varphi) \quad \forall\varphi \in F. \tag{3.3.3}$$

Proof. Let $\varphi \in F$, then due to assumption (i), the mapping $u \mapsto a(u,\varphi)$ is continuous in H. By the theorem of Riesz-Fréchet there is a unique $K\varphi \in H$ such that $\langle u, K\varphi\rangle_H = a(u,\varphi)$ for all $u \in H$. Moreover, the map $K\colon F \to H$, $\varphi \mapsto K\varphi$ is injective. Indeed, if $K\varphi = 0$, it follows

$$\alpha \left\|\varphi\right\|_F^2 \le |a(\varphi,\varphi)| = \langle K\varphi, \varphi\rangle_H = 0$$

and thus $\varphi = 0$. Consequently, $K\colon F \to \mathcal{R}(K)$ is bijective. Let $\varphi \in F$ so that $K\varphi = \psi \in \mathcal{R}(K)$, then

$$\alpha \left\|\varphi\right\|_F^2 \le |a(\varphi,\varphi)| = |\langle \varphi, K\varphi\rangle_H| \le \left\|\varphi\right\|_H \left\|K\varphi\right\|_H \le C \left\|\varphi\right\|_F \left\|K\varphi\right\|_H$$

for some constant $C > 0$, since $F \hookrightarrow H$ continuously. This shows that the inverse function $K^{-1}\colon \mathcal{R}(K) \to F$ is bounded. We may therefore uniquely extend K^{-1} on the closure of $\mathcal{R}(K)$ with respect to $\left\|\cdot\right\|_H$ with values in the completion $\tilde{F}$ of F with respect to $\left\|\cdot\right\|_F$. Let $L \in F'$ then L is uniquely continuously extendable to $\tilde{F}$. Let $\xi \in \tilde{F}$ such that

$$\langle \varphi, \xi\rangle_F = L(\varphi)$$

for all $\varphi \in \tilde{F}$. We denote by P the orthogonal projection on $\overline{\mathcal{R}(K)} \subset H$ with respect to $\langle\cdot,\cdot\rangle_H$ and define the map $K_P^{-1} = K^{-1}P\colon H \to \tilde{F}$. Let us denote by $(K_P^{-1})'\colon \tilde{F} \to H$ the adjoint operator. We claim that $u = (K_p^{-1})'\xi \in H$ is a solution of equation (3.3.3). Indeed, let $\varphi \in F$, then

$$a(u,\varphi) = \langle u, K\varphi\rangle_H = \left\langle (K_P^{-1})'\xi, K\varphi\right\rangle_H = \left\langle \xi, K_P^{-1}K\varphi\right\rangle_F = \langle \xi, \varphi\rangle_F = L(\varphi).$$

□

Remark 3.3.5. We note that theorem 3.3.4 does not make any statement about the uniqueness of the solution u. It holds that u is unique if and only if $\mathcal{R}(K)$ is dense in H. In fact, if $\mathcal{R}(K)$ is dense and $u, v \in H$ are two solutions, it follows that $\langle u - v, K\varphi\rangle = 0$ for all $\varphi \in F$ and therefore $u = v$. Moreover, if we know that the solution is unique and suppose that $\mathcal{R}(K)$ is not dense in H, then there is $0 \ne v \in \overline{\mathcal{R}(K)}^{\perp} \subset H$. If u is the unique solution of equation (3.3.3), then $u + v$ is a solution, too.

Theorem 3.3.6. *For every $T > 0$, $h \in L^2((0,T)\times\Omega\times\mathbb{R}^n)$, $g \in L^2(\Sigma_-^T)$ and $f \in L^2(\Omega\times\mathbb{R}^n)$ there exists a weak solution of* (3.3.2)*.*

Proof. We are going to apply theorem 3.3.4. We choose the Hilbert space H as defined in equation (3.3.2). To ensure the coercivity of the bilinear form a, we need to consider the problem for $\lambda > 0$ first. We define the subspace $F = \{\varphi \in C_c^\infty(Q_T) \mid \varphi = 0 \text{ on } \Sigma_+^T\} \subset H$.

On F we define the prehilbertian norm

$$\|\varphi\|_F^2 = \|\varphi\|_H^2 + \frac{1}{2}\|\varphi\|_{L^2(\Sigma_-^T)}^2 + \frac{1}{2}\|\varphi(0,\cdot)\|_{L^2(\Omega\times\mathbb{R}^n)}^2.$$

Clearly, the injection of F into H is continuous. We define the bilinear form $a\colon H \times F \to \mathbb{R}$ as

$$a(u,\varphi) = \int_{Q_T} -u\partial_t\varphi - uv\cdot\nabla_x\varphi + \lambda u\varphi + \sigma\langle\nabla_v u, \nabla_v\varphi\rangle\,\mathrm{d}v\mathrm{d}x\mathrm{d}t.$$

Let $\varphi \in F$, then $a(\,\cdot,\varphi)$ is clearly continuous H, since the norm of H controls u and $\nabla_v u$ for every $u \in H$. Let $\varphi \in F$, then

$$\begin{aligned} a(\varphi,\varphi) &= \int_{Q_T} -\varphi\partial_t\varphi - \varphi(v\cdot\nabla_x\varphi) + \lambda\varphi^2 + \sigma\langle\nabla_v\varphi,\nabla_v\varphi\rangle\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\ &= -\int_{Q_T}\varphi\partial_t\varphi\mathrm{d}v\mathrm{d}x\mathrm{d}t - \int_{Q_T}\varphi(v\cdot\nabla_x\varphi)\mathrm{d}v\mathrm{d}x\mathrm{d}t + \int_{Q_T}\left(\lambda\varphi^2 + \sigma\left|\nabla_v\varphi\right|^2\right)\mathrm{d}v\mathrm{d}x\mathrm{d}t. \end{aligned}$$

The last term can be controlled from below by $\min(\lambda,\sigma)\|\varphi\|_H^2$ so that we have to deal with the two remaining terms. Partial integration with respect to t shows

$$-\int_{Q_T}\varphi\partial_t\varphi\mathrm{d}v\mathrm{d}x\mathrm{d}t = \int_{Q_T}(\partial_t\varphi)\varphi\mathrm{d}v\mathrm{d}x\mathrm{d}t + \int_{\Omega\times\mathbb{R}^n}\varphi(0,x,v)^2\mathrm{d}v\mathrm{d}x.$$

Applying the divergence theorem in the x variable shows that

$$-\int_{Q_T}\varphi v\cdot\nabla_x\varphi\mathrm{d}v\mathrm{d}x\mathrm{d}t = \int_{Q_T}(v\cdot\nabla_x\varphi)\varphi\mathrm{d}v\mathrm{d}x\mathrm{d}t - \int_{\Sigma_+^T\cup\Sigma_-^T}\varphi^2\langle v, n(x)\rangle\,\mathrm{d}S(x)\mathrm{d}v\mathrm{d}t$$

and consequently

$$-\int_{Q_T}\varphi(v\cdot\nabla_x\varphi)\mathrm{d}\mathrm{d}v\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_{\Sigma_-^T}\varphi^2\left|\langle v,n(x)\rangle\right|\mathrm{d}S(x)\mathrm{d}v\mathrm{d}t,$$

since in the surface integral only $x \in \partial\Omega$ such that $\pm\langle v,n(x)\rangle > 0$ are of relevance and $\varphi = 0$ on Σ_+^T. The last three equations combined show that $a(\varphi,\varphi) \geq \min(1,\lambda,\sigma)\|\varphi\|_F^2$. Thus, we can apply theorem 3.3.4. Let us define the linear functional $L\colon F \to \mathbb{R}$ by

$$L(\varphi) = \int_{Q_T} h\varphi\mathrm{d}v\mathrm{d}x\mathrm{d}t + \int_{\Omega\times\mathbb{R}^n} f\varphi(0,x,v)\mathrm{d}v\mathrm{d}x + \int_{\Sigma_-^T} g\varphi\left|\langle v,n(x)\rangle\right|\mathrm{d}S(x)\mathrm{d}v\mathrm{d}t.$$

Using the Cauchy-Schwarz inequality, we see that $L \in F'$. By theorem 3.3.4, there exists $u \in H$ such that $a(u,\varphi) = L(\varphi)$ for all $\varphi \in F$. Finally, since the weak formulation of $h + \sigma\Delta_v u - \lambda u$ defines a linear functional on H, we deduce $\partial_t u + v\cdot\nabla_x u \in \mathcal{T}$ and thus u

is a weak solution of equation (3.3.2). If $\lambda = 0$, we rescale the boundary values g and h by $\exp(-\alpha t)$ for some $\alpha > 0$ and solve the equation

$$\partial_t u + v \cdot \nabla_x u + \alpha u = \sigma \Delta_v u + \exp(-\alpha t) h \tag{3.3.4}$$

with initial datum f and boundary values $\exp(-\alpha t) g$. Denote by v the corresponding solution, then $u = \exp(\lambda t) v$ solves the original problem as can be seen by a change of variable in the integral equation for the weak solution. □

The next step is to understand if the traces of a solution u on Σ_-^T are well-defined and whether u attains the initial value f. A result towards this question was given in [Car98, Proposition 2.4]. Unfortunately, there is an error in its proof. The flaw and the explicit problem are described in [AM19]. We refer to the article [AM19] for further information on this thematic.

Remark 3.3.7. The homogenous boundary value problem (3.3.2) on the interval $[0, 1]$ and the regularity of solutions is also studied in the article [HJV14].

4 L^p-spectrum of Kolmogorov equations with constant coefficients

We want to examine the spectral properties of the Kolmogorov equation in $L^p(\mathbb{R}^N)$. For $p \in (1, \infty)$ we investigate the operator $(\mathcal{K}_p, D(\mathcal{K}_p))$ and the respective semigroup $T(t)f$ as defined in (3.1.14). For the reader's convenience we recall the definition of $(\mathcal{K}_p, D(\mathcal{K}_p))$:

$$\mathcal{K}_p u = \operatorname{div}(A\nabla u) + \langle x, B\nabla u\rangle$$

has to be understood in the distributional sense for all $u \in D(\mathcal{K}_p) = \{u \in L^p(\mathbb{R}^N) \mid \mathcal{K}_p u \in L^p(\mathbb{R}^N)\}$. We are going to make the structural assumptions presented in section 3.1 on the matrices $A, B \in \mathbb{R}^{N\times N}$. It turns out that it is helpful to investigate spectral properties of the drift operator, given, in the distributional sense, by

$$\mathcal{B}_p u = \langle x, B\nabla u\rangle \tag{4.0.1}$$

for all $u \in D(\mathcal{B}_p) = \{u \in L^p(\mathbb{R}^N) \mid \mathcal{B}_p u \in L^p(\mathbb{R}^N)\}$ first. We are going to write $\mathcal{B}u$ if the differentiation can be understood in the classical sense.

Lemma 4.0.1. The drift operator $(\mathcal{B}_p, D(\mathcal{B}_p))$ is a closed operator in $L^p(\mathbb{R}^N)$.

Proof. To show that $\mathcal{B}_p$ is closed, let $(u_n)_{n\in\mathbb{N}} \subset D(\mathcal{B}_p)$ be a sequence converging to u such that the sequence $(\mathcal{B}_p u_n)_{n\in\mathbb{N}}$ converges to g in $L^p(\mathbb{R}^N)$. Given any $\varphi \in C_c^\infty(\mathbb{R}^N)$, we calculate

$$\begin{aligned}\int_{\mathbb{R}^N} g\varphi \mathrm{d}x &= \lim_{n\to\infty}\int_{\mathbb{R}^N} \mathcal{B}_p u_n g \mathrm{d}x = \lim_{n\to\infty}\sum_{i,j=1}^{n}\int_{\mathbb{R}^N} \varphi x_i b_{ij}\partial_{x_j} u_n \mathrm{d}x \\ &= -\lim_{n\to\infty}\sum_{i,j=1}^{n}\int_{\mathbb{R}^N} \delta_{ij} b_{ij} u_n\varphi + x_i b_{ij} u_n \partial_{x_j}\varphi \mathrm{d}x \\ &= -\operatorname{tr}(B)\int_{\mathbb{R}^N} u\varphi \mathrm{d}x - \int_{\mathbb{R}^N} u\mathcal{B}\varphi \mathrm{d}x = -\int_{\mathbb{R}^N} u\mathcal{B}\varphi \mathrm{d}x,\end{aligned}$$

since $x_i\varphi \in C_c^\infty(\mathbb{R}^N)$ for all $i = 1, \dots, N$. We conclude $u \in D(\mathcal{B}_p)$ and $\mathcal{B}_p u = g \in L^p(\mathbb{R}^N)$ and therefore that the drift operator is closed in $L^p(\mathbb{R}^N)$. □

Proposition 4.0.2. The operator $(\mathcal{B}_p, D(\mathcal{B}_p))$ is the generator of the isometric C_0-group $(S(t))_{t\in\mathbb{R}}$ defined as

$$[S(t)f](x) = f(\exp(tB^T)x)$$

for all $f \in L^p(\mathbb{R}^N), x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Further, $C_c^\infty(\mathbb{R}^N)$ is a core of $\mathcal{B}_p$.

Proof. In the proof of theorem 3.1.25 we have already seen that $S(t)$ defines a strongly continuous group on $L^p(\mathbb{R}^N)$. Thus, it remains to characterize its generator. We first show that $C_c^\infty(\mathbb{R}^N)$ is a core of the generator $(G_p, D(G_p))$ of $S(t)$. Clearly, $C_c^\infty(\mathbb{R}^N)$ is an invariant subspace of $S(t)$ and $C_c^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$. Let $f \in C_c^\infty(\mathbb{R}^N)$, then for arbitrary $x \in \mathbb{R}^N$ we have

$$\lim_{t\to 0} \frac{[S(t)f](x) - f(x)}{t} = \langle x, B\nabla f(x)\rangle.$$

Since f is of compact support and thus there is an $R > 0$ such that for all $t \in (0, 1)$ it holds $\operatorname{supp} S(t)f \subset B_R(0)$, we deduce that this convergence holds in $L^p(\mathbb{R}^n)$ as well. Consequently, $f \in D(G_p)$ and $G_p f = \mathcal{B}_p f$. Proposition A.1.1 implies that $(\mathcal{B}, C_c^\infty(\mathbb{R}^N))$ is a core of $(D(G_p), G_p)$. By lemma 4.0.1, the drift operator is closed so that we deduce $D(G_p) \subset D(\mathcal{B}_p)$ and $G_p u = \mathcal{B}_p u$ for all $u \in D(G_p)$. To show that $D(G_p) = D(\mathcal{B}_p)$, we are going to use duality of $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$, where $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We denote by $(D(G_q), G_q)$ the generator of $S(t)$ in $L^q(\mathbb{R}^N)$. It holds

$$\int_{\mathbb{R}^N} \mathcal{B}_p u\varphi \mathrm{d}x = -\int_{\mathbb{R}^N} u\langle x, B\nabla\varphi\rangle \mathrm{d}x = -\int_{\mathbb{R}^N} uG_q\varphi \mathrm{d}x$$

for all $\varphi \in C_c^\infty(\mathbb{R}^N)$. Since G_q is closed and thus $C_c^\infty(\mathbb{R}^N)$ is dense in $D(G_q)$ with respect to the graph norm, we deduce

$$\int_{\mathbb{R}^N} \mathcal{B}_p u\varphi \mathrm{d}x = -\int_{\mathbb{R}^N} uG_q\varphi \mathrm{d}x$$

for all $\varphi \in D(G_q)$. We recall that, since $S(t)$ is a group of isometries, we know that $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \neq 0\} \subset \rho(G_p)$. Let $u \in D(\mathcal{B}_p)$, then there is a $\lambda \in \rho(G_p)$ with $-\lambda \in \rho(G_q)$ and $v \in D(G_p) \subset D(\mathcal{B}_p)$ such that $\lambda u - \mathcal{B}_p u = \lambda v - G_p v$. We define $w = v - u \in D(\mathcal{B}_p)$, then $\lambda w - \mathcal{B}_p w = 0$ and thus

$$0 = \int_{\mathbb{R}^N} (\lambda w - \mathcal{B}_p w)\varphi \mathrm{d}x = -\int_{\mathbb{R}^N} w(\lambda + G_q)\varphi \mathrm{d}x$$

for all $\varphi \in D(G_q)$. Due to the fact that $-\lambda \in \rho(G_q)$, we see that $(\lambda + G_q)(D(G_q)) = L^q(\mathbb{R}^N)$

and so we deduce $w = 0$ by duality of $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$. This shows $u \in D(G_p)$ and therefore $D(\mathcal{B}_p) = D(G_p)$. $\square$

Lemma 4.0.3. Let $0 \neq B \in \mathbb{R}^{N\times N}$ be a matrix satisfying the structural assumption from section 3.1. There exists an open subset $U \subset \mathbb{R}^N$ such that $\lim\limits_{|t|\to\infty} \left|\exp(tB^T)x\right| = \infty$ uniformly in x on every compact subset of U.

Proof. The matrix B is nilpotent of order r, i.e. $B^{r+1} = 0$ and $B^r \neq 0$. We choose

$$U = \{x \in \mathbb{R}^N \mid (B^T)^r x \neq 0\}.$$

Let $K \subset U$ be a compact set and $x \in K$, then by the Cauchy-Schwarz inequality and the Peter-Paul inequality with parameter $\varepsilon = \frac{1}{2}$, we deduce

$$\begin{aligned}\left|\exp(tB^T)x\right|^2 = \left|\sum_{k=0}^{r} \frac{t^k}{k!}(B^T)^k x\right|^2 &= \frac{1}{2(r!)^2} t^{2r} \left|(B^T)^r x\right|^2 - \left|\sum_{k=0}^{r-1} \frac{t^k}{k!}(B^T)^k x\right|^2 \\ &\geq \frac{1}{2(r!)^2} t^{2r} \left|(B^T)^r x\right|^2 - c_1 t^{2r-2}\end{aligned}$$

for a constant $c_1 = c_1(K) > 0$ and all $t > 1$. This shows the uniform divergence, since due to the compactness of $K \subset U$ it holds $\min_{x\in K} \left|(B^T)^r x\right|^2 > 0$. $\square$

Theorem 4.0.4. *For all $p \in (1,\infty)$ the spectrum of $(\mathcal{B}_p, D(\mathcal{B}_p))$ is given by $\sigma(\mathcal{B}_p) = i\mathbb{R}$.*

Proof. Since $S(t)$ is a group of isometries, we immediately know that $\sigma(\mathcal{B}_p) \subset i\mathbb{R}$. Suppose that there exists $\alpha \in \mathbb{R}$ such that $\alpha i \in \rho(\mathcal{B}_p)$, then there is a $\delta > 0$ such that $B_\delta(\alpha i) \subset \rho(\mathcal{B}_p)$. Let $\varepsilon > 0$ and $f \in L^p(\mathbb{R}^N)$, then by the Laplace formula for the resolvent, we have

$$\begin{aligned}R(\varepsilon + ai, \mathcal{B}_p)f &= \int_0^\infty \exp(-\varepsilon t - iat)S(t)f\,\mathrm{d}t \\ R(-\varepsilon + ai, \mathcal{B}_p)f &= -\int_0^\infty \exp(-\varepsilon t + iat)S(-t)f\,\mathrm{d}t\end{aligned}$$

for all $|a - \alpha| < \delta$. Setting

$$V(\varepsilon + ai)f = R(\varepsilon + ai, \mathcal{B}_p)f - R(-\varepsilon + ai, \mathcal{B}_p)f = \int_{-\infty}^\infty \exp(-\varepsilon\,|t| - ita)S(t)f\,\mathrm{d}t,$$

it follows that $\lim\limits_{\varepsilon\to 0} V(\varepsilon + ia)f = 0$ for all $|a - \alpha| < \delta$ and every $f \in L^p(\mathbb{R}^N)$. According to proposition 4.0.3, there exists an open set $U \subset \mathbb{R}^N$ such that $\left|\exp(tB^T)x\right| \to \infty$ uniformly on every compact subset as $t \to \infty$. We pick any nonnegative $0 \neq f \in C_c^\infty(U)$ and define the function $g\colon \mathbb{R} \to \mathbb{R}$ as

$$g(t) = \int_{\mathbb{R}^N} [S(t)f](x)f(x)\mathrm{d}x$$

for every $t \in \mathbb{R}$. It holds $g \in C^\infty(\mathbb{R})$ and since $\left|\exp(tB^T)x\right| \to \infty$ for $t \to \infty$ uniformly over the compact set $\operatorname{supp} f$, we further deduce that $g \in C_c^\infty(\mathbb{R})$. By Fubini's theorem it holds

$$\int_{\mathbb{R}^N} [V(\varepsilon + ia)f](x)f(x)\mathrm{d}x = \int_{-\infty}^{\infty} \exp(-\varepsilon\,|t| - iat)g(t)\mathrm{d}t.$$

Using the fact that f and g are of compact support and that $\lim_{\varepsilon\to 0} V(\varepsilon + ia) = 0$, we deduce $\hat{g}(a) = 0$ for all $|a - \alpha| < \delta$ using the theorem of dominated convergence. As the Fourier transform of a compactly supported smooth function $\hat{g}$ is real analytic, we conclude that $g = 0$. This is a contradiction to $g(0) = \int_{\mathbb{R}^N} f^2\mathrm{d}x > 0$. We conclude $\sigma(\mathcal{B}_p) = i\mathbb{R}$. ☐

We want to study the boundary spectrum of $\mathcal{K}$. To do so, we need the following two theorems. The first one goes back to Hale Trotter and Tosio Kato and gives a criteria to show strong convergence of a sequence of semigroups. The second one is taken from [DS86] and considers the spectrum of certain limits of semigroups.

Theorem 4.0.5. *For $k \in \mathbb{N}$ let $(T(t))_{t\geq 0}$ and $(T_k(t))_{t\geq 0}$ be strongly continuous semigroups on a Banach space X with generators A, A_k, respectively. We assume that there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\|, \|T_k(t)\| \leq M\exp(\omega t)$$

for all $t \geq 0$ and every $k \in \mathbb{N}$. Let D_0 be a core of $(A, D(A))$ and assume that $D_0 \subset D(A_k)$ for every $k \in \mathbb{N}$ as well as that $A_k x \to Ax$ for every $x \in D_0$. Then $T_k(t)x \to T(t)x$ for all $x \in X$ uniformly in t on compact intervals.

Proof. This is a special case of [EN00, III Theorem 4.8]. ☐

Theorem 4.0.6. *Let $S(t)$ and $T(t)$ be strongly continuous semigroups on a Banach space X with generators B and A. Assume that there exists a sequence of invertible isometries $V_k\colon X \to X$ such that $V_k^{-1}T(t)V_k x \to S(t)x$ for all $x \in X$ uniformly on compact intervals. If $S(t)$ is a group of isometries, then $\sigma(A) \subset \sigma(B)$.*

Proof. [DS86, Corollary 13] ☐

Theorem 4.0.7. *For all $p \in (1,\infty)$ it holds $i\mathbb{R} \subset \sigma(\mathcal{K}_p) \subset \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \leq 0\}$.*

Proof. For every $k \in \mathbb{N}$ we define the linear operator $V_k \colon L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ as $[V_k f](x) = k^{-\frac{N}{p}} f(k^{-1}x)$ for any $f \in L^p(\mathbb{R}^N)$ and all $x \in \mathbb{R}^N$. A simple substitution in the integral shows that V_k is an isometry. Given any $f \in C_c^\infty(\mathbb{R}^N)$, it holds

$$V_k^{-1} \mathcal{K} V_k f = \frac{1}{k^2} \operatorname{div}(A \nabla f) + \langle x, B \nabla u \rangle. \tag{4.0.2}$$

The latter equation shows that $V_k^{-1} \mathcal{K} V_k f \to \mathcal{B} f$ in $L^p(\mathbb{R}^N)$ for $k \to \infty$. Recalling the behavior of semigroups and their generators under isometries, theorem 4.0.5 shows that $V_k^{-1} T(t) V_k$ converges strongly to $S(t)$. Using theorem 4.0.6, we conclude $i\mathbb{R} = \sigma(\mathcal{B}_p) \subset \sigma(\mathcal{K}_p)$. By theorem 3.1.25, it holds $\sigma(\mathcal{K}_p) \subset \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \leq 0\}$ and hence we deduce the claim. □

Remark 4.0.8. Let us give an interpretation of the technique that has been used in the previous proof. From the physical point of view one could say that the isometries $(V_k)_{k \in \mathbb{N}}$ zoom out preserving the physical quantity of mass. Zooming out far enough, only the drift term determines the behavior of the equation. After zooming in again, only the effect of the drift term remains significant.

Corollary 4.0.9. The Kolmogorov semigroup $(T(t))_{t \geq 0}$ on $L^p(\mathbb{R}^N)$ is not analytic.

Proof. The spectrum of an analytic semigroup cannot contain any vertical lines in $\mathbb{C}$. Since $i\mathbb{R}$ is contained in the spectrum for all $p \in (1, \infty)$, the Kolmogorov semigroup cannot be analytic. □

Corollary 4.0.10. The growth bound of $(T(t))_{t \geq 0}$ can be calculated as $\omega(T) = 0$.

Proof. We know that $(T(t))_{t \geq 0}$ is a positive semigroup on $L^p(\mathbb{R}^N)$ and that $s(\mathcal{K}) = 0$. By the theorem in [Wei95], it must hold $\omega(T) = s(\mathcal{K}) = 0$. □

Corollary 4.0.11. For all $p \in (1, \infty)$ and all $t \geq 0$ it holds $\|T(t)\| = 1$.

Proof. If it were $\|T(t)\| < 1$ for some $t > 0$, it must hold $\omega(T) < 0$ by [EN00, Chapter V, Proposition 1.7]. This is a contradiction to $\omega(T) = 0$. □

5 Long-time behavior of Kolmogorov equations with constant coefficients

In this section we are going to investigate the behavior of solutions to a Kolmogorov equation with constant coefficients for long times briefly. More precisely, we are going to investigate equations as presented in section 3.1

$$\partial_t u = \operatorname{div}(A\nabla u) + \langle x, B\nabla u\rangle .$$

Theorem 5.0.1. *Let $f \in L^1(\mathbb{R}^N)$ with mean $M = \int_{\mathbb{R}^N} f \mathrm{d}x$. It holds*

$$\|T(t)f - M\Gamma(t,x)\|_{1,\mathbb{R}^N} \to 0$$

as $t \to \infty$. If f has finite expectation, i.e. $xf(x) \in L^1(\mathbb{R}^N)$, then there exists a constant $C = C(n, A, B)$ such that

$$\|T(t)f - M\Gamma(t,x)\|_{1,\mathbb{R}^N} \le Ct^{-\frac{1}{2}}\|xf(x)\|_{1,\mathbb{R}^N}$$

for all $t > 0$.

Remark 5.0.2. This theorem and its proof are based on an analog theorem for the heat equation, which can be found for example in [QS07][Proposition 48.6]. We note that above theorem recovers the classical statement in the case of $A = \mathrm{Id}_N$ and $B = 0$.

Proof. Let $f \in L^1(\mathbb{R}^N)$ such that $xf(x) \in L^1(\mathbb{R}^N)$. Given $x \in \mathbb{R}^N$ and $t > 0$, using the mean value theorem, we calculate

$$\begin{aligned}
&[T(t)f](x) - M\Gamma(t,x) \\
&= \frac{c_0}{t^{\frac{Q}{2}}} \int_{\mathbb{R}^N} \left[\exp\left(-\frac{1}{4}\left\langle \mathcal{C}^{-1}(t)(x - E(t)y), x - E(t)y\right\rangle\right) - \exp\left(-\frac{1}{4}\left\langle \mathcal{C}^{-1}(t)x, x\right\rangle\right)\right] f(y)\mathrm{d}y \\
&= \frac{c_0}{2t^{\frac{Q}{2}}} \int_{\mathbb{R}^N}\int_0^1 \left\langle \delta_{\frac{1}{\sqrt{t}}} E(t)y, \mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t}}}\alpha\right\rangle \exp\left(-\frac{1}{4}\left\langle \mathcal{C}^{-1}(t)\alpha, \alpha\right\rangle\right) f(y)\mathrm{d}\theta\mathrm{d}y
\end{aligned}$$

abbreviating $\alpha = x - \theta E(t)y$. For $t > 1$, using lemma 3.1.9 and the fact that $\mathcal{C}^{-1}(1)$ is positive definite with constant $\lambda > 0$, we deduce

$$
\begin{aligned}
&|[T(t)f](x) - M\Gamma(t,x)| \\
&\le \frac{c_1}{t^{\frac{Q}{2}}} \int_{\mathbb{R}^N} \int_0^1 \left|\delta_{\frac{1}{\sqrt{t}}} E(t)y\right| \left|\delta_{\frac{1}{\sqrt{t}}}\alpha\right| \exp\left(-\frac{\lambda}{2}\left|\delta_{\frac{1}{\sqrt{t}}}\alpha\right|^2\right) \exp\left(-\frac{1}{2}\left\langle \mathcal{C}^{-1}(t)\alpha,\alpha\right\rangle\right) |f(y)| \,\mathrm{d}\theta \mathrm{d}y \\
&\le \frac{c_2}{t^{\frac{Q}{2}}} \int_{\mathbb{R}^N} \int_0^1 \left|\delta_{\frac{1}{\sqrt{t}}} E(t)y\right| \exp\left(-\frac{1}{2}\left\langle \mathcal{C}^{-1}(t)\alpha,\alpha\right\rangle\right) |f(y)| \,\mathrm{d}\theta \mathrm{d}y
\end{aligned}
$$

for new constants $c_1, c_2 > 0$, since it holds $\sup_{s\ge 0} s\exp\left(-\frac{\lambda}{2}s^2\right) < \infty$. Integrating with respect to x and using the theorem of Fubini leads to

$$
\begin{aligned}
&\|T(t)f - M\Gamma(t,x)\|_{1,\mathbb{R}^N} \\
&\le c_2 \int_{\mathbb{R}^N} \int_0^1 \left|\delta_{\frac{1}{\sqrt{t}}} E(t)y\right| |f(y)| \int_{\mathbb{R}^N} t^{-\frac{Q}{2}} \exp\left(-\frac{1}{2}\left\langle C^{-1}(t)\alpha,\alpha\right\rangle\right) \mathrm{d}x\mathrm{d}\theta\mathrm{d}y \\
&\le c_3 \int_{\mathbb{R}^N} \int_0^1 \left|\delta_{\frac{1}{\sqrt{t}}} E(t)y\right| |f(y)| \,\mathrm{d}\theta\mathrm{d}y = c_3 \int_{\mathbb{R}^N} \left|\delta_{\frac{1}{\sqrt{t}}} E(t)y\right| |f(y)| \,\mathrm{d}y \\
&\le \frac{c_4}{t^{\frac{1}{2}}} \int_{\mathbb{R}^N} |y|\,|f(y)|\,\mathrm{d}y = \frac{c_4}{t^{\frac{1}{2}}} \|xf(x)\|_{1,\mathbb{R}^N}
\end{aligned}
$$

for some constants $c_3, c_4 > 0$ depending only on dimension and $\mathcal{C}^{-1}(1)$. We have used lemma 3.1.8, i.e. $\left|\delta_{\frac{1}{\sqrt{t}}} E(t)y\right| \le t^{-\frac{1}{2}} c(B)\,|y|$ and that

$$
\frac{1}{t^{\frac{Q}{2}}} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2}\left\langle \mathcal{C}^{-1}(t)(x-\theta E(t)y), x-\theta E(t)y\right\rangle\right) \mathrm{d}x
$$

is a constant independent of θ and t. This can be seen by incorporating the factor $\frac{1}{2}$ into a new diffusion matrix $\tilde{A} = \frac{1}{2}A$. After the substitution $z = \theta E(t)x$ this integral can be written as the integral of the fundamental solution to $\tilde{A}$ and B multiplied by a constant. Lemma 3.1.14 applied to this new fundamental shows that this integral is equal to a constant for all points $\theta E(t)y$. This shows the second part of the statement.

To prove the first statement, we choose a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^N)$ with mean M approximating f in $L^1(R^N)$. Then clearly, $x\varphi_k(x) \in L^1(\mathbb{R}^N)$ and thus using the second statement together with the contractivity of $T(t)$, we conclude

$$
\begin{aligned}
\|T(t)f - M\Gamma(t,x)\|_{1,\mathbb{R}^N} &\le \|T(t)f - T(t)\varphi_k\|_{1,\mathbb{R}^N} + \|T(t)\varphi_k - M\Gamma(t,x)\|_{1,\mathbb{R}^N} \\
&\le \|f-\varphi_k\|_{1,\mathbb{R}^N} + \|T(t)\varphi_k - M\Gamma(t,x)\|_{1,\mathbb{R}^N}
\end{aligned}
$$

and therefore $\limsup_{t\to\infty} \|T(t)f - M\Gamma(t,x)\|_{1,\mathbb{R}^N} \le \|f-\varphi_k\|_{1,\mathbb{R}^N}$, which shows the conclusion as $k\to\infty$. □

6 Regularity of Kolmogorov equations

6.1 C^∞-regularity of Kolmogorov equations with constant coefficients

We want to apply the regularity theory of hypoelliptic differential operators, presented in section 2.4, to the setting of Kolmogorov equations with constant coefficients. We consider the partial differential equation

$$\partial_t u = \operatorname{div}(A\nabla u) + \langle x, B\nabla u\rangle,$$

where A, B satisfy the structural assumptions given in section 3.1. At first, we need to show that our equation can be written as a sum of squares of first order homogenous differential operators as in equation (2.4.1). We introduce the matrix $Q = (q_{ij}) = A^{\frac{1}{2}}$ which is the square root of the diffusion matrix A. Let us define the vector fields $X_i = \sum_{j=1}^{m_0} q_{ij}\partial_{x_j} = [Q\nabla u]_i$ for $i = 1, \dots, m_0$. It holds

$$\sum_{i=1}^{m_0} X_i^2 u = \sum_{i=1}^{m_0}\sum_{j=1}^{m_0} q_{ij}\partial_{x_j}\sum_{k=1}^{m_0} q_{ik}\partial_{x_k} u = \sum_{i,j,k=1}^{m_0} q_{ij}q_{ik}[\nabla^2 u]_{jk} = \operatorname{tr}(QQ\nabla^2 u) = \operatorname{div}(A\nabla u)$$

so that, if we define

$$X_0 = \langle x, B\nabla\rangle - \partial_t,$$

we get

$$\sum_{j=1}^{m_0} X_j^2 u + X_0 u = \operatorname{div}(A\nabla u) + \langle x, B\nabla u\rangle - \partial_t u.$$

This shows that we may be able to apply the theorem of Hörmander if the Lie condition in theorem 2.4.7 is satisfied.

Theorem 6.1.1. *Let $\Omega \subset \mathbb{R}^N$ be open, $T > 0$ and $h \in C^\infty((0,T)\times\Omega)$. Every distributional solution of the equation*

$$\partial_t u = \operatorname{div}(A\nabla u) + \langle x, B\nabla u\rangle + h \text{ in } (0,T)\times\Omega$$

is smooth in the interior of its domain in time and space.

Remark 6.1.2. In particular, theorem 6.1.1 applies to distributional solutions of either equation (3.1.1) or equation (3.3.2) with $h \in C^\infty(Q_T)$.

We note that theorem 6.1.1 does not immediately show that the weak solutions of equation (3.3.2), obtained in section 3.3.2, are smooth. This is due to the fact that the notion of weak solution for this equation may be different from a distributional solution. However, by restricting the set of all test functions in the definition 3.3.2 to $C_c^\infty(\mathring{Q}_T)$ it follows that a weak solution is also a distributional solution and therefore by theorem 6.1.1 is smooth in the interior of its domain.

Proof. We have shown that the Kolmogorov equation can be written in the form of equation (2.4.1). We need to verify the Hörmander bracket condition (2.4.2). To do so, we calculate the Lie bracket

$$\begin{aligned}
[X_i, \langle x, B\nabla\rangle] = [(Q\nabla)_i, \langle x, B\nabla\rangle] &= \sum_{j=1}^{m_0} q_{ij}\partial_{x_j}\langle x, B\nabla\rangle - \sum_{j=1}^{m_0}\langle x, B\nabla\rangle q_{ij}\partial_{x_j} \\
&= \sum_{j=1}^{m_0} q_{ij}\partial_{x_j} \sum_{k,l=1}^{N} x_k b_{kl}\partial_{x_l} - \sum_{j=1}^{m_0}\langle x, B\nabla\rangle q_{ij}\partial_{x_j} \\
&= \sum_{j=1}^{m_0} q_{ij} \sum_{k,l=1}^{N} \delta_{jk} b_{kl}\partial_{x_l} + \sum_{j=1}^{m_0}\sum_{k,l=1}^{N} b_{kl}x_k\partial_{x_l} q_{ij}\partial_{x_j} - \sum_{j=1}^{m_0}\langle x, B\nabla\rangle q_{ij}\partial_{x_j} \\
&= \sum_{j=1}^{m_0} q_{ij}\sum_{l=1}^{N} b_{jl}\partial_{x_l} = [QB\nabla]_i
\end{aligned}$$

for $i = 1, \ldots, m_0$. For $j \geq 0$ and $i = 1, \ldots, m_0$ we define

$$X_{i,j+1} = [X_{i,j}, \langle x, B\nabla\rangle]$$

and set $X_{i,0} = X_i$. We have seen that $X_{i,1} = [QB\nabla]_i$ for every $i = 1, \ldots, m_0$ and claim that it holds

$$X_{i,j} = [QB^j\nabla u]_i$$

for every $j \geq 0$ and any $i = 1, \ldots, m_0$. We prove this claim for every $i = 1, \ldots, m_0$ by induction. The base case has already been proven. Assume that the claim is true for some $j \geq 0$, then

$$X_{i,j+1} = [X_{i,j}, \langle x, B\nabla\rangle] = [[QB^j\nabla]_i, \langle x, B\nabla\rangle] = \sum_{k,l=1}^{N}(QB^j)_{ik} b_{kl}\partial_{x_l} = [QB^{j+1}\nabla]_i$$

by a calculation similar to that in the base case. Thus, by the principle of induction it is $X_{i,j} = [QB^j\nabla u]_i$ for all $j \geq 0$.

Next, we claim that the dimension of the Lie algebra generated by

$$V = \{X_{i,j} \mid i = 1, \ldots, m_0,\ j = 1, \ldots, r\} \cup X_0$$

is given as $N+1$ at every point of $\mathbb{R}^{N+1}$. We note that $[QB^j\nabla]_i$ is the i-th row of $QB_1 \cdots B_j[\nabla u]^{(j)}$ and that $QB_1 \cdots B_j \in \mathbb{R}^{m_0 \times m_j}$ is of rank $m_j \leq m_0$. Therefore, it holds that the vectors $[QB^j\nabla]_1, \ldots, [QB^j\nabla]_{m_0}$, i.e. the columns of $QB_1 \cdots B_j$, contain a generator of the subspace $\{(t,x) \in \mathbb{R}^{N+1} \mid x^{(l)} = 0,\ l \neq j\} \subset \mathbb{R}^N$ for all $j = 0, \ldots, r$. Furthermore, ∂_t is a basis vector of the time component. We conclude that the dimension must be equal to $N+1$. Consequently, the vector fields $X_0, \ldots, X_{m_0}$ satisfy the assumptions of Hörmander's theorem and we conclude the proof of our theorem. □

Remark 6.1.3. The interior regularity of solutions to the Kolmogorov equation on $\mathbb{R}^N$ can also be derived from the representation of solutions in terms of the fundamental solution. However, this does not work for the Kolmogorov equation posed on arbitrary subsets $\Omega \subset \mathbb{R}^N$. This is due to the fact that on an arbitrary set Ω there must not be a fundamental solution.

6.2 Maximal L^p-regularity of Kolmogorov equations

This section is devoted to study maximal L^p-regularity of the Kolmogorov equation with constant coefficients. We recall the definition of maximal L^p-regularity.

Definition 6.2.1. Let $p \in (1,\infty)$ and $A\colon D(A) \to X$ be a densely defined and closed linear operator on a Banach space X. We say that A is of maximal L^p-regularity if there exists a constant $C > 0$ such that for all $f \in L^p((0,\infty;X))$ there is a unique solution $u \in L^p(0,\infty;D(A))$ to the Cauchy problem

$$\begin{cases} [\partial_t u](t) = [Au](t) + f(t), & \text{for almost every } t > 0 \\ u(0) = 0, \end{cases}$$

which also implies that $\partial_t u \in L^p((0,\infty);X)$ and the estimate

$$\|u\|_{p,(0,\infty)} + \|\partial_t u\|_{p,(0,\infty)} + \|Au\|_{p,(0,\infty)} \leq C\|f\|_{p,(0,\infty)}. \tag{6.2.1}$$

Unfortunately, every realization of $(\mathcal{K}_q, D(\mathcal{K}_q))$ in some $L^q(\mathbb{R}^N)$ space is not of maximal L^p-regularity for every $p \in (1,\infty)$. This is a consequence of the following property of operators with maximal L^p-regularity and corollary 4.0.9.

Proposition 6.2.2. Assume that $A\colon D(A) \to X$ is an operator of maximal L^p-regularity on some Banach space X, then A generates a bounded analytic semigroup.

Proof. [EW09, Proposition 2.2 in the section Maximal regularity and applications to PDEs] □

Corollary 6.2.3. Let $q \in (1, \infty)$ and consider the $L^q(\mathbb{R}^N)$ realization of the Kolmogorov operator $\mathcal{K}_q$. Then $\mathcal{K}$ is not of maximal L^p-regularity for every $p \in (1, \infty)$.

Remark 6.2.4. We want to recall that the non-analyticity of $\mathcal{K}$ is a consequence of the fact that the spectrum of the drift part of $\mathcal{K}$ is the imaginary line. It is well known that elliptic second order differential operators with constant coefficients are of maximal L^p-regularity. Therefore, there is some reason to blame the drift term for the loss of regularity in this situation. In the next section we are going to investigate this matter. Moreover, we are going to see how much regularity in x in terms of a Sobolev norm we can get at most.

6.3 Hypoelliptic kinetic regularity

We have already seen multiple times that the characteristic properties of the Kolmogorov equation are the elliptic diffusion in the velocity variable and the coupling of velocity with the position in the transport term. In particular, it is not a good property of the Kolmogorov equation that there is no elliptic diffusion in the x variable. In chapter 2.2 we have artificially added some diffusion in x to obtain existence results. Moreover, in section 6.1 it was shown that the coupling via the transport term, measured in the commutator

$$[\partial_{v_i}, v \cdot \nabla_x] = \partial_{x_i}, \tag{6.3.1}$$

leads to interior regularity which is as good as the regularity in the elliptic case. In this section we want to investigate whether a version of the estimate (6.2.1) holds for the Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \sigma \Delta_v u + f \\ u(0) = 0 \end{cases}$$

in $L^2(\mathbb{R}^{2n+1})$. Our aim is to show estimates in fractional Sobolev norms. To be more precise, we are going to show that one gains $\frac{1}{3}$ of a derivative in the position variable. This section is based on the article [Bou02] written by François Bouchut. Fractional derivatives and fractional Sobolev Spaces are discussed in the appendix A.2. In the following we are going to write $D_x^\beta = (-\Delta_x)^{\frac{\beta}{2}}$ and $D_v^\beta = (-\Delta_v)^{\frac{\beta}{2}}$.

Let us consider the kinetic transport equation

$$\partial_t u + v \cdot \nabla_x u = h$$

first. Our aim is to show a gain of differentiability in x under the assumption of additional regularity of u in the velocity variable.

Theorem 6.3.1. *Let $\beta \geq 0$. We assume that $u, h \in L^2(\mathbb{R}^{2n+1})$ satisfy*

$$\partial_t u + v \cdot \nabla_x u = h$$

in the distributional sense. If additionally

$$D_v^\beta u \in L^2(\mathbb{R}^{2n+1}),$$

then $D_x^{\frac{\beta}{1+\beta}} u \in L^2(\mathbb{R}^{2n+1})$ and

$$\left\| D_x^{\frac{\beta}{1+\beta}} u \right\|_{2,\mathbb{R}^N} \leq C \left\| D_v^\beta u \right\|_{2,\mathbb{R}^N}^{\frac{1}{1+\beta}} \left\| h \right\|_{2,\mathbb{R}^N}^{\frac{\beta}{1+\beta}},$$

where $C = C(\beta, n) \geq 0$.

Proof. We are going to use Fourier transform in the variables (t, x). The Fourier variables of (t, x) are denoted by (ξ, k) and the Fourier transform by

$$\mathcal{F}_{t,x} u(\xi, k, v) = \frac{1}{(2\pi)^{\frac{n+2}{2}}} \int_{\mathbb{R}^N} \exp(-it\xi - i\langle k, x\rangle) u(t, v, x) \mathrm{d}x \mathrm{d}t.$$

Since $\partial_t u + v \cdot \nabla_x u = h \in L^2(\mathbb{R}^{2n+1})$, we deduce by the theorem of Fubini that it holds $\partial_t u(\cdot, v, \cdot) + v \cdot \nabla_x u(\cdot, v, \cdot) = h(\cdot, v, \cdot) \in L^2(\mathbb{R}^{n+1})$ for almost every $v \in \mathbb{R}^n$ and hence

$$i(\xi + \langle v, k\rangle)[\mathcal{F}_{t,x} u](\xi, k, v) = [\mathcal{F}_{t,x} h](\xi, k, v) \tag{6.3.2}$$

for almost every $(\xi, k, v) \in \mathbb{R}^{2n+1}$. By $C(\beta, n)$, we are going to denote a constant depending on the differentiability β and on dimension n which can change from line to line. Let us consider a mollifier $\omega \in C_c^\infty(\mathbb{R}^n)$ as in section A.3 and the corresponding Dirac sequence $(\omega_\varepsilon)_{\varepsilon>0}$. Additionally, we assume that

$$\int_{\mathbb{R}^n} v^\alpha \omega(v) \mathrm{d}v = 0 \tag{6.3.3}$$

holds for all multi-indices $\alpha \in \mathbb{N}^n$ of length $1 \leq |\alpha| < \beta$. Such a mollifier can be constructed for example by subtracting the orthogonal projection on a suitable subspace generated by the respective monomials. Moreover, we assume that $\omega \le C(\beta, n)$. We fix $(\xi, k) \in \mathbb{R}^{n+1}$ and write

$$[\mathcal{F}_{t,x}u](\xi,k,v) = (\omega_\varepsilon * \mathcal{F}_{t,x}(\xi,k,\cdot))(v) + [[\mathcal{F}_{t,x}u](\xi,k,v) - (\omega_\varepsilon * [\mathcal{F}_{t,x}u](\xi,k,\cdot))(v)] = I_1 + I_2.$$

We are going to handle the latter two terms separately. To estimate the first term, let $\lambda > 0$. We add $\lambda \mathcal{F}_{t,x}u$ to both sides of equation (6.3.2) and get

$$\lambda[\mathcal{F}_{t,x}u](\xi,k,v) + i(\xi + \langle v,k\rangle)[\mathcal{F}_{t,x}u](\xi,k,v) = \lambda[\mathcal{F}_{t,x}u](\xi,k,v) + [\mathcal{F}_{t,x}h](\xi,k,v).$$

Dividing by $\lambda + i(\xi + \langle k,v\rangle)$ leads to

$$\begin{aligned}[\mathcal{F}_{t,x}u](\xi,k,v) &= \frac{\lambda[\mathcal{F}_{t,x}u](\xi,k,v) + [\mathcal{F}_{t,x}h](\xi,k,v)}{\lambda + i(\xi + \langle k,v\rangle)} \\ &= \frac{[\mathcal{F}_{t,x}u](\xi,k,v) + \frac{1}{\lambda}[\mathcal{F}_{t,x}h](\xi,k,v)}{1 + \frac{i}{\lambda}(\xi + \langle k,v\rangle)}\end{aligned}$$

and smoothing in velocity with ω_ε in turn gives

$$(\omega_\varepsilon * [\mathcal{F}_{t,x}u](\xi,k,\cdot))(v) = \int_{\mathbb{R}^n} \frac{[\mathcal{F}_{t,x}u](\xi,k,y) + \frac{1}{\lambda}[\mathcal{F}_{t,x}h](\xi,k,y)}{1 + \frac{i}{\lambda}(\xi + \langle k,y\rangle)} \omega_\varepsilon(v-y)\mathrm{d}y.$$

Using the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}|(\omega_\varepsilon * [\mathcal{F}_{t,x}u](\xi,k,\cdot))(v)| &\le \int_{\mathbb{R}^N} \left| \frac{[\mathcal{F}_{t,x}u](\xi,k,y) + \frac{1}{\lambda}[\mathcal{F}_{t,x}h](\xi,k,y)}{1 + \frac{i}{\lambda}(\xi + \langle k,y\rangle)} \right| \left(|\omega_\varepsilon(v-y)|^{\frac{1}{2}} \right)^2 \mathrm{d}y \\ &\le \left\| \left| [\mathcal{F}_{t,x}u](\xi,k,\cdot) + \frac{1}{\lambda}[\mathcal{F}_{t,x}h](\xi,k,\cdot) \right| |\omega_\varepsilon(v-\cdot)|^{\frac{1}{2}} \right\|_{2,\mathbb{R}^n} \\ &\quad \cdot \left(\int_{\mathbb{R}^n} \frac{|\omega_\varepsilon(v-y)|}{\left|1 + \frac{i}{\lambda}(\xi + \langle k,y\rangle)\right|^2} \mathrm{d}y \right)^{\frac{1}{2}}. \end{aligned} \tag{6.3.4}$$

By orthogonal decomposition every $y \in \mathbb{R}^n$ can be written as $y = y'\frac{k}{|k|} + \tilde{y}$ where $y' = \langle y, \frac{k}{|k|}\rangle$ and $\langle \tilde{y}, k\rangle = 0$. From $\omega \le C(\beta, n)$ we deduce that

$$|\omega_\varepsilon(v)| \le \varepsilon^{-n} C(\beta,n) \mathbb{1}_{\{|v| \le \varepsilon\}}$$

for all $v \in \mathbb{R}^n$. We write

$$v - y = (v' - y')\frac{k}{|k|} + \tilde{v} - \tilde{y},$$

whence

$$\{|v-y|\le\varepsilon\}\subset\{|v'-y'|\le\varepsilon\}\cap\{|\tilde v-\tilde y|\le\varepsilon\}.$$

Performing the transformation which corresponds to the basis change in the representation of y and v, we get

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{|\omega_\varepsilon(v-y)|}{\left|1+\frac{i}{\lambda}(\xi+\langle k,y\rangle)\right|^2}\mathrm{d}y &\le C(\beta,n)\frac{1}{\varepsilon^n}\int_{B_\varepsilon(v)}\frac{1}{\left|1+\frac{i}{\lambda}(\xi+\langle k,y\rangle)\right|^2}\mathrm{d}y\\ &\le C(\beta,n)\frac{1}{\varepsilon^n}\int_{v'-\varepsilon}^{v'+\varepsilon}\frac{1}{\left|1+\frac{i}{\lambda}(\xi+y'\,|k|)\right|^2}\int_{B_\varepsilon(\tilde v)}\mathrm{d}\tilde y\mathrm{d}y'\\ &\le C(\beta,n)\frac{\lambda^2}{\varepsilon\,|k|}\int_{-\infty}^{\infty}\frac{1}{\lambda^2+z^2}\mathrm{d}z=C(\beta,n)\frac{\lambda}{\varepsilon\,|k|}.\end{aligned}$$

Using the latter estimate in inequality (6.3.4) shows

$$|(\omega_\varepsilon*[\mathcal{F}_{t,x}u](\xi,k,\cdot))(v)|^2\le\frac{\lambda C(\beta,n)}{\varepsilon\,|k|}\left\|\left|[\mathcal{F}_{t,x}u](\xi,k,\cdot)+\frac{1}{\lambda}[\mathcal{F}_{t,x}h](\xi,k,\cdot)\right|\,|\omega_\varepsilon(v-\cdot)|^{\frac12}\right\|_{2,\mathbb{R}^n}^2.$$

Eventually, integrating with respect to the velocity v and taking the square root, we get

$$\begin{aligned}&\|(\omega_\varepsilon*[\mathcal{F}_{t,x}u](\xi,k,\cdot))\|_{2,\mathbb{R}^N}\\ &\le\left(\frac{\lambda C(\beta,n)}{\varepsilon\,|k|}\right)^{\frac12}\int_{\mathbb{R}^n}\left\|\left|[\mathcal{F}_{t,x}u](\xi,k,\cdot)+\frac{1}{\lambda}[\mathcal{F}_{t,x}h](\xi,k,\cdot)\right|\,|\omega_\varepsilon(v-\cdot)|^{\frac12}\right\|_{2,\mathbb{R}^n}\mathrm{d}v\\ &=\left(\frac{\lambda C(\beta,n)}{\varepsilon\,|k|}\right)^{\frac12}\left\|\left|[\mathcal{F}_{t,x}u](\xi,k,\cdot)+\frac{1}{\lambda}[\mathcal{F}_{t,x}h](\xi,k,\cdot)\right|\right\|_{2,\mathbb{R}^n}\\ &\le\left(\frac{\lambda C(\beta,n)}{\varepsilon\,|k|}\right)^{\frac12}\left[\|[\mathcal{F}_{t,x}u](\xi,k,\cdot)\|_{2,\mathbb{R}^n}+\frac{1}{\lambda}\|[\mathcal{F}_{t,x}h](\xi,k,\cdot)\|_{2,\mathbb{R}^n}\right]\end{aligned}$$

by the theorem of Fubini and the triangle inequality. Choosing

$$\lambda=\frac{\|[\mathcal{F}_{t,x}h](\xi,k,\cdot)\|_{2,\mathbb{R}^n}}{\|[\mathcal{F}_{t,x}u](\xi,k,\cdot)\|_{2,\mathbb{R}^n}},$$

we deduce

$$\|(\omega_\varepsilon*[\mathcal{F}_{t,x}u](\xi,k,\cdot))\|_{2,\mathbb{R}^N}\le\frac{C(\beta,n)}{\sqrt{\varepsilon\,|k|}}\|[\mathcal{F}_{t,x}u](\xi,k,\cdot)\|_{2,\mathbb{R}^N}^{\frac12}\|[\mathcal{F}_{t,x}h](\xi,k,\cdot)\|_{2,\mathbb{R}^N}^{\frac12}.$$

The second term can be estimated as

$$\|I_2\|_{2,\mathbb{R}^n} = \|[\mathcal{F}_{t,x}u](\xi,k,\cdot) - (\omega_\varepsilon * [\mathcal{F}_{t,x}u](\xi,k.\cdot))\|_{2,\mathbb{R}^n} \le C(\beta,n)\varepsilon^\beta \left\| D_v^\beta [\mathcal{F}_{t,x}u](\xi,k,\cdot)\right\|_{2,\mathbb{R}^n}.$$

This estimate follows immediately from the estimate

$$\left|1-\sqrt{2\pi}^n[\mathcal{F}_v\omega_\varepsilon](\eta)\right| \le C(\beta,N)\,|\varepsilon\eta|^\beta \tag{6.3.5}$$

for all $\eta \in \mathbb{R}^N$ and the theorem of Plancherel for Fourier transformation in the velocity variable. Indeed, it holds

$$\begin{aligned}
\|[\mathcal{F}_{t,x}u](\xi,k,\cdot) - (\omega_\varepsilon * [\mathcal{F}_{t,x}u](\xi,k.\cdot))\|_{2,\mathbb{R}^n}^2 &= \|\mathcal{F}_v\left([\mathcal{F}_{t,x}u](\xi,k,\cdot) - (\omega_\varepsilon * [\mathcal{F}_{t,x}u](\xi,k.\cdot))\right)\|_{2,\mathbb{R}^n}^2 \\
&= \left\|\mathcal{F}_v\left([\mathcal{F}_{t,x}u](\xi,k,\cdot)\right)\left(1-\sqrt{2\pi}^n\mathcal{F}_v(\omega_\varepsilon)\right)\right\|_{2,\mathbb{R}^n}^2 \\
&\le C(\beta,n)^2\varepsilon^{2\beta}\left\||\eta|^\beta\left[\mathcal{F}_v\left([\mathcal{F}_{t,x}u](\xi,k,\cdot)\right)\right](\eta)\right\|_{2,\mathbb{R}^n}^2 \\
&= C(\beta,n)^2\varepsilon^{2\beta}\left\|D_v^\beta[\mathcal{F}_{t,x}u](\xi,k,\cdot)\right\|_{2,\mathbb{R}^n}^2.
\end{aligned}$$

For every $\beta \in \mathbb{N}$ the inequality in equation (6.3.5) can be proven by using a Taylor expansion of the exponential function in the integrand. The vanishing moments condition, as assumed in equation (6.3.3), shows that all lower order terms are zero. For a more rigorous reasoning we refer to [Bur13, Section 2.5]. The case $0 < \beta \in \mathbb{R}\setminus\mathbb{N}$ follows by an interpolation argument. To be more precise, we choose $\theta \in (0,1)$ such that $\beta = (1-\theta)\lceil\beta\rceil + \theta\lfloor\beta\rfloor$. Due to the fact that the vanishing moment condition was assumed for all multi-indices $\alpha \in \mathbb{N}^n$ of length $1 \le |\alpha| < \beta$, it clearly holds true for all multi-indices $\alpha \in \mathbb{N}^n$ of length $1 \le |\alpha| < \lceil\beta\rceil$, too. This shows that

$$\begin{aligned}
\left|1-\sqrt{2\pi}^n[\mathcal{F}_v\omega_\varepsilon](\eta)\right| &\le C(\beta,N)\,|\varepsilon\eta|^{\lceil\beta\rceil} \\
\left|1-\sqrt{2\pi}^n[\mathcal{F}_v\omega_\varepsilon](\eta)\right| &\le C(\beta,N)\,|\varepsilon\eta|^{\lfloor\beta\rfloor},
\end{aligned}$$

whence

$$\begin{aligned}
\left|1-\sqrt{2\pi}^n[\mathcal{F}_v\omega_\varepsilon](\eta)\right| &= \left|1-\sqrt{2\pi}^n[\mathcal{F}_v\omega_\varepsilon](\eta)\right|^{1-\theta+\theta} \le C(\beta,n)\,|\varepsilon\eta|^{(1-\theta)\lceil\beta\rceil}\,|\varepsilon\eta|^{\theta\lfloor\beta\rfloor} \\
&= C(\beta,n)\,|\varepsilon\eta|^\beta
\end{aligned}$$

for all $\eta \in \mathbb{R}^n$.

Together the estimates on I_1 and I_2 yield

$$\begin{aligned}\|[\mathcal{F}_{t,x}u](\xi,k,\cdot)\|_{2,\mathbb{R}^n} \leq \frac{C(\beta,n)}{\sqrt{\varepsilon\,|k|}}\,\|[\mathcal{F}_{t,x}u](\xi,k,\cdot)\|_{2,\mathbb{R}^N}^{\frac{1}{2}}\,\|[\mathcal{F}_{t,x}h](\xi,k,\cdot)\|_{2,\mathbb{R}^N}^{\frac{1}{2}}\\ + C(\beta,n)\varepsilon^{\beta}\left\|D_v^{\beta}\mathcal{F}_{t,x}u(\xi,k,\cdot)\right\|_{2,\mathbb{R}^n}\end{aligned}$$

for all $\varepsilon > 0$. We want to choose $\varepsilon > 0$ such that the mixed term on the right-hand side can be absorbed into the left-hand side. A possible choice is that of

$$\varepsilon = \frac{4C(\beta,n)^2\,\|[\mathcal{F}_{t,x}h](\xi,k,\cdot)\|_{2,\mathbb{R}^N}}{|k|\,\|[\mathcal{F}_{t,x}u](\xi,k,\cdot)\|_{2,\mathbb{R}^N}}.$$

After multiplication by $2\,\|[\mathcal{F}_{t,x}u](\xi,k,\cdot)\|_{2,\mathbb{R}^N}^{\beta}$, we get

$$\|[\mathcal{F}_{t,x}u](\xi,k,\cdot)\|_{2,\mathbb{R}^N}^{\beta+1} \leq 2C(\beta,n)\frac{1}{|k|^{\beta}}\left\|D_v^{\beta}\mathcal{F}_{t,x}u(\xi,k,\cdot)\right\|_{2,\mathbb{R}^n}\|[\mathcal{F}_{t,x}h](\xi,k,\cdot)\|_{2,\mathbb{R}^N}^{\beta},$$

whence

$$\left\||k|^{\frac{\beta}{\beta+1}}\,[\mathcal{F}_{t,x}u](\xi,k,\cdot)\right\|_{2,\mathbb{R}^N} \leq C(\beta,n)\left\|D_v^{\beta}\mathcal{F}_{t,x}u(\xi,k,\cdot)\right\|_{2,\mathbb{R}^n}^{\frac{1}{1+\beta}}\|[\mathcal{F}_{t,x}h](\xi,k,\cdot)\|_{2,\mathbb{R}^N}^{\frac{\beta}{1+\beta}}.$$

Squaring and integrating in (ξ,k) yields

$$\begin{aligned}\left\|D_x^{\frac{\beta}{\beta+1}}u\right\|_{2,\mathbb{R}^{2n+1}}^2 &\leq C(\beta,n)\int_{\mathbb{R}^{n+1}}\left\|D_v^{\beta}\mathcal{F}_{t,x}u(\xi,k,\cdot)\right\|_{2,\mathbb{R}^n}^{\frac{2}{1+\beta}}\|[\mathcal{F}_{t,x}h](\xi,k,\cdot)\|_{2,\mathbb{R}^N}^{\frac{2\beta}{1+\beta}}\,\mathrm{d}(\xi,k)\\ &\leq C(\beta,n)\left\|D_v^{\beta}u\right\|_{2,\mathbb{R}^{2n+1}}\|h\|_{2,\mathbb{R}^N}^{\frac{\beta}{\beta+1}}\end{aligned}$$

by the Hölder inequality with exponent $\beta+1$ and $\frac{\beta+1}{\beta}$ as well as the theorem of Plancherel. □

Remark 6.3.2. Let us note that in the previous proof the estimate of the second term can be seen as an estimate characterizing the rate of convergence of the approximation. It is only due to the fact that $D_v^{\beta}u \in L^2(\mathbb{R}^{2n+1})$ that we obtain the rate of β. One can say that in some sense this rate of convergence quantifies the regularity properties of the function u.

Theorem 6.3.3. *Let $u \in C_c^{\infty}(\mathbb{R}^{2n+1})$ be a solution of*

$$\partial_t u + v\cdot\nabla_x u = \sigma\Delta_v u + h$$

for a function $h \in C_c^{\infty}(\mathbb{R}^{2n+1})$ and some $\sigma > 0$. In this case it is

$$\|\partial_t u + v\cdot\nabla_x u\|_{2,\mathbb{R}^{2n+1}} + \sigma\,\|\Delta_v u\|_{2,\mathbb{R}^{2n+1}} \leq C\,\|h\|_{2,\mathbb{R}^{2n+1}}$$

for some constant $C \geq 0$ independent of $\sigma > 0$. In particular, it holds $D_x^{\frac{2}{3}} u \in L^2(\mathbb{R}^{2n+1})$ as well as

$$\left\| D_x^{\frac{2}{3}} u \right\|_{2,\mathbb{R}^{2n+1}} \leq \frac{C}{\sigma^{\frac{1}{3}}} \|h\|_{2,\mathbb{R}^{2n+1}} .$$

Proof. Applying the fractional derivative of order $\frac{1}{3}$ in the space variable to both sides of the Kolmogorov equation, we obtain

$$(\partial_t + v \cdot \nabla_x - \sigma \Delta_v) D_x^{\frac{1}{3}} u = D_x^{\frac{1}{3}} h.$$

Multiplying by $D_x^{\frac{1}{3}} u$ and integrating on $\mathbb{R}^{2n+1}$ leads to

$$\sigma \left\| \nabla_v \left(D_x^{\frac{1}{3}} u \right) \right\|_{2,\mathbb{R}^{2n+1}} = \sigma \langle \nabla_v D_x^{\frac{1}{3}} u, \nabla_v D_x^{\frac{1}{3}} u \rangle = \langle D_x^{\frac{1}{3}} u, D_x^{\frac{1}{3}} h \rangle,$$

since

$$\int_{\mathbb{R}^{2n+1}} D_x^{\frac{1}{3}} u (\partial_t D_x^{\frac{1}{3}} u + v \cdot \nabla_x D_x^{\frac{1}{3}} u) \mathrm{d}x = 0.$$

This is due to the fact that in the t and v variables the function $D_x^{\frac{1}{3}} u$ is of compact support and infinitely often differentiable. Moreover, using the decay estimates from [DPP19, Chapter 1, Section 2, Proposition 2.9] for the fractional Laplacian of a Schwartz function, one can show that $\int (D_x^{\frac{1}{3}} u) v \cdot \nabla_x (D_x^{\frac{1}{3}} u) \mathrm{d}v \mathrm{d}x \mathrm{d}t = 0$.

We deduce

$$\sigma \left\| \nabla_v \left(D_x^{\frac{1}{3}} u \right) \right\|_{2,\mathbb{R}^{2n+1}}^2 = \langle D_x^{\frac{1}{3}} u, D_x^{\frac{1}{3}} h \rangle = \langle D_x^{\frac{2}{3}} u, h \rangle \leq \left\| D_x^{\frac{2}{3}} u \right\|_{2,\mathbb{R}^{2n+1}} \|h\|_{2,\mathbb{R}^{2n+1}}$$

by Lemma A.2.3. Multiplying by $-\Delta_v u$ and integrating over $\mathbb{R}^{2n+1}$ shows

$$- \langle \Delta_v u, v \cdot \nabla_x u \rangle + \sigma \|\Delta_v u\|_{2,\mathbb{R}^{2n+1}}^2 = -\langle \Delta_v u, h \rangle \tag{6.3.6}$$

and furthermore

$$\begin{aligned} \langle \Delta_v u, v \cdot \nabla_x u \rangle &= \sum_{i=1}^n \langle \partial_{v_i}^2 u, v \cdot \nabla_x u \rangle \\ &= -\sum_{i=1}^n \langle \partial_{v_i} u, \partial_{v_i} (v \cdot \nabla_x u) \rangle \\ &= -\sum_{i=1}^n \langle \partial_{v_i} u, v \cdot \nabla_x (\partial_{v_i} u) + \partial_{x_i} u \rangle. \end{aligned}$$

Due to

$$\langle \partial_{v_i} u, v \cdot \nabla_x (\partial_{v_i} u) \rangle = 0$$

for all $i = 1, \dots, n$, we deduce

$$\langle \Delta_v u, v \cdot \nabla_x u \rangle = -\langle \nabla_v u, \nabla_x u \rangle. \tag{6.3.7}$$

Combining the equations (6.3.6) and (6.3.7), arguing as in lemma A.2.4 and using the estimate from equation (6.3) leads to

$$\begin{aligned}
\sigma \|\Delta_v u\|^2_{2,\mathbb{R}^{2n+1}} &= \langle -\Delta_v u, h \rangle - \langle \nabla_v u, \nabla_x u \rangle \\
&\leq \|\Delta_v u\|_{2,\mathbb{R}^{2n+1}} \|h\|_{2,\mathbb{R}^{2n+1}} + \left\| \nabla_v \left(D_x^{\frac{1}{3}} u \right) \right\|_{2,\mathbb{R}^{2n+1}} \left\| D_x^{\frac{2}{3}} u \right\|_{2,\mathbb{R}^{2n+1}} \\
&\leq \|\Delta_v u\|_{2,\mathbb{R}^{2n+1}} \|h\|_{2,\mathbb{R}^{2n+1}} + \frac{1}{\sqrt{\sigma}} \left\| D_x^{\frac{2}{3}} u \right\|^{\frac{3}{2}}_{2,\mathbb{R}^{2n+1}} \|h\|^{\frac{1}{2}}_{2,\mathbb{R}^{2n+1}} .
\end{aligned} \tag{6.3.8}$$

Applying theorem 6.3.1 to the solution u of the kinetic equation with the right-hand side $\sigma \Delta_v u + h$ and $\beta = 2$, we deduce

$$\left\| D_x^{\frac{2}{3}} u \right\|_{2,\mathbb{R}^{2n+1}} \leq C \|\Delta_v u\|^{\frac{1}{3}}_{2,\mathbb{R}^{2n+1}} \|\partial_t u + v \cdot \nabla_x u\|^{\frac{2}{3}}_{2,\mathbb{R}^{2n+1}}$$

for some constant $C > 0$ independent of $\sigma > 0$. The Kolmogorov equation gives the estimate

$$\|\partial_t u + v \cdot \nabla_x u\|_{2,\mathbb{R}^{2n+1}} \leq \|h\|_{2,\mathbb{R}^{2n+1}} + \sigma \|\Delta_v u\|_{2,\mathbb{R}^{2n+1}} ,$$

whence

$$\begin{aligned}
\left\| D_x^{\frac{2}{3}} u \right\|^{\frac{3}{2}}_{2,\mathbb{R}^{2n+1}} &\leq C \|\Delta_v u\|^{\frac{1}{2}}_{2,\mathbb{R}^{2n+1}} \|\partial_t u + v \cdot \nabla_x u\|_{2,\mathbb{R}^{2n+1}} \\
&\leq C \|\Delta_v u\|^{\frac{1}{2}}_{2,\mathbb{R}^{2n+1}} \left(\|h\|_{2,\mathbb{R}^{2n+1}} + \sigma \|\Delta_v u\|_{2,\mathbb{R}^{2n+1}} \right) .
\end{aligned}$$

Combining the latter estimate with the estimate (6.3.8) shows

$$\begin{aligned}
&\sigma \|\Delta_v u\|^2_{2,\mathbb{R}^{2n+1}} \\
&\leq \|\Delta_v u\|_{2,\mathbb{R}^{2n+1}} \|h\|_{2,\mathbb{R}^{2n+1}} + \frac{1}{\sqrt{\sigma}} \left\| D_x^{\frac{2}{3}} u \right\|^{\frac{3}{2}}_{2,\mathbb{R}^{2n+1}} \|h\|^{\frac{1}{2}}_{2,\mathbb{R}^{2n+1}} \\
&\leq \|\Delta_v u\|_{2,\mathbb{R}^{2n+1}} \|h\|_{2,\mathbb{R}^{2n+1}} + \frac{1}{\sqrt{\sigma}} C \|\Delta_v u\|^{\frac{1}{2}}_{2,\mathbb{R}^{2n+1}} \left(\|h\|_{2,\mathbb{R}^{2n+1}} + \sigma \|\Delta_v u\|_{2,\mathbb{R}^{2n+1}} \right) \|h\|^{\frac{1}{2}}_{2,\mathbb{R}^{2n+1}} .
\end{aligned}$$

Using the Peter-Paul inequality three times with parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ shows

$$\begin{aligned}
&\sigma \left\|\Delta_v u\right\|_{2,\mathbb{R}^{2n+1}}^2 \\
&\leq \left\|\Delta_v u\right\|_{2,\mathbb{R}^{2n+1}} \left\|h\right\|_{2,\mathbb{R}^{2n+1}} + \frac{C}{\sqrt{\sigma}} \left\|\Delta_v u\right\|_{2,\mathbb{R}^{2n+1}}^{\frac{1}{2}} \left\|h\right\|_{2,\mathbb{R}^{2n+1}}^{\frac{3}{2}} + C\sqrt{\sigma} \left\|\Delta_v u\right\|_{2,\mathbb{R}^{2n+1}}^{\frac{3}{2}} \left\|h\right\|_{2,\mathbb{R}^{2n+1}}^{\frac{1}{2}} \\
&\leq \left(\frac{\varepsilon_1}{2} + \frac{C}{4\sigma^2\varepsilon_2^4} + \frac{C\sqrt{\sigma}}{\varepsilon_3^{\frac{4}{3}}}\right) \left\|\Delta_v u\right\|_{2,\mathbb{R}^{2n+1}}^2 + \left(\frac{1}{2\varepsilon_1} + \frac{3C\varepsilon_2^{\frac{4}{3}}}{4} + \frac{C\sqrt{\sigma}\varepsilon_3^4}{4}\right) \left\|h\right\|_{2,\mathbb{R}^{2n+1}}^2 .
\end{aligned}$$

Choosing $\varepsilon_1 = \sigma$, $\varepsilon_2 = C^{\frac{1}{4}}\sigma^{-\frac{3}{4}}$ and $\varepsilon_3 = (8C)^{\frac{4}{3}}\sigma^{-\frac{3}{8}}$ shows

$$\sigma \left\|\Delta_v u\right\|_{2,\mathbb{R}^{2n+1}}^2 \leq \frac{7}{8}\sigma \left\|\Delta_v u\right\|_{2,\mathbb{R}^{2n+1}}^2 + \frac{1}{\sigma}\left(\frac{1}{2} + \frac{3C^{\frac{4}{3}}}{4} + 128C^4\right) \left\|h\right\|_{2,\mathbb{R}^{2n+1}}^2 ,$$

whence

$$\left\|\Delta_v u\right\|_{2,\mathbb{R}^{2n+1}}^2 \leq \frac{C}{\sigma^2} \left\|h\right\|_{2,\mathbb{R}^{2n+1}}^2$$

for some constant $C \geq 0$ independent of $\sigma > 0$.

Consequently, it holds

$$\left\|\partial_t u + v \cdot \nabla_x u\right\|_{2,\mathbb{R}^{2n+1}} + \sigma \left\|\Delta_v u\right\|_{2,\mathbb{R}^{2n+1}} \leq (2C+1) \left\|h\right\|_{2,\mathbb{R}^{2n+1}}$$

and

$$\left\|D_x^{\frac{2}{3}} u\right\|_{2,\mathbb{R}^{2n+1}} \leq \frac{C}{\sigma^{\frac{1}{3}}} \left\|h\right\|_{2,\mathbb{R}^{2n+1}}$$

by theorem 6.3.1. □

Remark 6.3.4. (i) The exponent $\frac{2}{3}$ appeared first in the article [RS76]. Linda Rothschild and Elias Stein were able to prove the same gain of regularity in the position variable but only in a local sense. We refer to [RS76, Paragraph 18, Theorem 18].

(ii) Global $L^p(\mathbb{R}^N)$-estimates for the more general equation $\partial_t u = \operatorname{div}(A\nabla u) + \langle x, B\nabla u\rangle$ are also proven in [BCLP10] with different techniques.

7 Harnack inequalities for Kolmogorov equations

7.1 Kolmogorov equations with constant coefficients

Once again we are going to investigate the partial differential equation

$$\partial_t u = \operatorname{div}(A\nabla u) + \langle x, B\nabla u\rangle,$$

introduced in section 3.1. Throughout the following section we are going to make the assumptions given in section 3.1 on the matrices A and B. The aim of this section is to prove a Harnack inequality for nonnegative solutions of this equation. An elegant way of proving a Harnack inequality is to prove a differential Harnack inequality first and to deduce the Harnack inequality from this differential inequality. This approach goes back to Peter Li and Shing Tung Yau, who have proven in [LY86] a Harnack inequality for the heat equation on a manifold by means of a differential Harnack inequality. This section follows the article [PP04a] closely.

7.1.1 The differential Harnack inequality

The aim of this section is to prove a differential Harnack inequality for nonnegative solutions of $\partial_t u = \mathcal{K}u$ on a strip $(0,T) \times \mathbb{R}^N$. We are going to use this differential Harnack inequality in section 7.1.2 to prove a Harnack inequality for the equation $\partial_t u = \mathcal{K}u$. We start with a differential equality for the fundamental solution Γ of $\mathcal{K}$, from which we are going to deduce the desired inequality.

Proposition 7.1.1. The fundamental solution Γ of $\mathcal{K}$ satisfies the gradient equation

$$-Y\Gamma + \frac{Q}{2t}\Gamma = \frac{\langle A\nabla\Gamma, \nabla\Gamma\rangle}{\Gamma}$$

in $(0,\infty) \times \mathbb{R}^N$. We recall that $Y = \langle x, B\nabla\rangle - \partial_t$.

Proof. By definition 3.1.12, we can write the fundamental solution at $t_2 = t$, $x_2 = x$ and $t_1 = 0$, $x_1 = 0$ as

$$\Gamma(t,x,0,0) = \Gamma(t,x) = c_o t^{-\frac{Q}{2}} \exp\left(-\frac{1}{4}\left\langle \mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t}}}x, \delta_{\frac{1}{\sqrt{t}}}x\right\rangle\right)$$

for all $t > 0$. We calculate the diffusive partial derivatives of first order

$$\partial_{x_i}\Gamma(t,x) = -\frac{1}{4}\Gamma(t,x)\partial_{x_i}\left\langle \mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t}}}x, \delta_{\frac{1}{\sqrt{t}}}x\right\rangle = -\frac{1}{2\sqrt{t}}\Gamma(t,x)(\mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t}}}x)_i$$

for $i = 1, \dots, m_0$ and these of second order

$$\begin{aligned}\partial_{x_i x_j}\Gamma(t,x) &= -\frac{1}{2\sqrt{t}}(\mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t}}}x)_j\partial_{x_i}\Gamma(t,x) - \frac{(\mathcal{C}^{-1}(1))_{ij}}{2t}\Gamma(t,x)\\ &= \frac{1}{4t}(\mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t}}}x)_i(\mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t}}}x)_j\Gamma(t,x) - \frac{(\mathcal{C}^{-1}(1))_{ij}}{2t}\Gamma(t,x)\end{aligned}$$

for all $i, j = 1, \dots, m_0$. Together we deduce

$$\begin{aligned}[\mathrm{div}(A\nabla\Gamma)](t,x) &= \sum_{i=1}^{m_0}\sum_{j=1}^{m_0} a_{ij}\partial_{x_i}\partial_{x_j}\Gamma(t,x)\\ &= \sum_{i,j=1}^{m_0} a_{ij}\left(\frac{1}{4t}(\mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t}}}x)_i(\mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t}}}x)_j\Gamma(t,x) - \frac{(\mathcal{C}^{-1}(1))_{ij}}{2t}\Gamma(t,x)\right)\\ &= \frac{1}{\Gamma(t,x)}\sum_{i,j=1}^{m_0} a_{ij}\left(\frac{1}{2\sqrt{t}}(\mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t}}}x)_i\Gamma(t,x)\frac{1}{2\sqrt{t}}(\mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t}}}x)_j\Gamma(t,x)\right)\\ &\quad - \frac{\Gamma(t,x)}{2t}\sum_{i,j=1}^{m_0} a_{ij}\left(\mathcal{C}^{-1}(1)\right)_{ij}\\ &= \frac{1}{\Gamma(t,x)}\sum_{i=1}^{m_0}[\partial_{x_i}\Gamma](t,x)\sum_{j=1}^{m_0} a_{ij}[\partial_{x_j}\Gamma](t,x) - \frac{\Gamma}{2t}\sum_{i,j=1}^{m_0} a_{ij}\left(\mathcal{C}^{-1}(1)\right)_{ij}\\ &= \frac{\langle A\nabla\Gamma, \nabla\Gamma\rangle(t,x)}{\Gamma(t,x)} - \frac{\Gamma}{2t}\sum_{i,j=1}^{m_0} a_{ij}\left(\mathcal{C}^{-1}(1)\right)_{ij}. \qquad (7.1.1)\end{aligned}$$

Furthermore, it is

$$\begin{aligned}[Y\Gamma](t,x) &= [\langle x, B\nabla\rangle\Gamma](t,x) - \partial_t\Gamma(t,x)\\ &= \Gamma(t,x)\left(\frac{Q}{2t} - \frac{1}{2}\langle x, B\mathcal{C}^{-1}(t)x\rangle + \frac{1}{4}\partial_t\langle \mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t}}}x, \delta_{\frac{1}{\sqrt{t}}}x\rangle\right) \qquad (7.1.2)\end{aligned}$$

using above representation of $\mathcal{C}(t)$. We evaluate the equation $[\mathcal{K}\Gamma](t,x) = 0$ at $x = 0$. Clearly, this is equivalent to the equation $[\mathrm{div}(A\nabla\Gamma)](t,0) = -[Y\Gamma](t,0)$. Applying equation (7.1.1) and equation (7.1.2) gives

$$\frac{\Gamma(t,x)}{2t}\sum_{i,j=1}^{m_0} a_{ij}(\mathcal{C}^{-1}(1))_{ij} = \frac{Q}{2t}\Gamma(t,x),$$

taking into account that $\partial_t\langle\mathcal{C}^{-1}(1)\delta_{\frac{1}{\sqrt{t}}}x, \delta_{\frac{1}{\sqrt{t}}}x\rangle\Big|_{x=0} = 0$. Finally, using $[\mathcal{K}\Gamma](t,x) = 0$, the above representation of Q and equation (7.1.1), we conclude

$$-[Y\Gamma](t,x) + \frac{Q}{2t} = [\mathrm{div}(A\nabla\Gamma)](t,x) + \frac{Q}{2t} = \frac{\langle A\nabla\Gamma, \nabla\Gamma\rangle(t,x)}{\Gamma(t,x)}.$$

This shows the claim if $x_1 = 0$ and $t_1 = 0$. It holds that

$$\Gamma(t-t_1, x - E(t-t_1)x_1) = \Gamma(t-t_1, x-E(t-t_1)x_1, 0, 0) = \Gamma(t,x,t_1,x_1).$$

We want to evaluate the previously partially proven gradient equation (7.1.1) at $(t-t_1, x-E(t-t_1)x_1, 0, 0)$. However, we have to take into account that $x - E(t-t_1)x_1$ is now time-dependent. We have already seen that this is no problem, since we know $[Y\Gamma](t-t_1, x-E(t-t_1)x_1, 0, 0) = Y\Gamma(t-t_1, x-E(t-t_1)x_1, 0, 0)$ from remark 3.1.7 . $\square$

Remark 7.1.2. In the following theorem we want to interchange differentiation with integration without clear justification. To apply the well known measure theoretic theorem, which allows such an interchange, we would need an integrable majorant of $\nabla\Gamma(t,x,t_0,\cdot)u(t_0,\cdot)$. However, we only know that u is nonnegative. Therefore, we are going to assume from now on that $u \geq 0$ is a nonnegative solution such that this interchange of differentiation and integration is permitted. This is for example the case if u is polynomially bounded.

Theorem 7.1.3. *Let $T > 0$ and let u be a positive solution of $\partial_t u = \mathcal{K}u$ in $[0,T]\times\mathbb{R}^N$. In this case u satisfies the differential Harnack inequality*

$$-Yu + \frac{Q}{2t}u \geq \frac{\langle A\nabla u, \nabla u\rangle}{u} \tag{7.1.3}$$

in the whole strip $(0,T)\times\mathbb{R}^N$.

Proof. As announced in the introduction of this section, this is a consequence of proposition (7.1.1) along with the representation of the positive solution u in terms of the fundamental solution as presented in theorem 3.1.24. To be more precise, given $t_0 \in (0,T)$, the function u can be written as

$$u(t,x) = \int_{\mathbb{R}^N}\Gamma(t,x,t_0,y)u(t_0,y)\mathrm{d}y$$

for all $(t,x) \in (t_0,T) \times \mathbb{R}^N$. Using proposition 7.1.1 and interchanging differentiation and integration, we get

$$\begin{aligned} -Yu + \frac{Q}{2t}u &= \int_{\mathbb{R}^N} \left(-Y\Gamma(\cdot,t_0,y) + \frac{Q}{2t}\Gamma(\cdot,\cdot,t_0,y) \right) u(t_0,y)\mathrm{d}y \\ &= \int_{\mathbb{R}^N} \left(\frac{\langle A\nabla\Gamma(\cdot,\cdot,t_0,y), \nabla\Gamma(\cdot,\cdot,t_0,y)\rangle}{\Gamma(\cdot,\cdot,t_0,y)} \right) u(t_0,y)\mathrm{d}y. \end{aligned}$$

Applying lemma B.0.4 and interchanging the order of integration and differentiation, we conclude from

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(\frac{\langle A\nabla\Gamma(\cdot,\cdot,t_0,y), \nabla\Gamma(\cdot,\cdot,t_0,y)\rangle}{\Gamma(\cdot,\cdot,t_0,y)} \right) u(t_0,y)\mathrm{d}y \int_{\mathbb{R}^N} \Gamma(\cdot,\cdot,t_0,y)u(t_0,y)\mathrm{d}y \\ &\geq \int_{\mathbb{R}^N} \langle A\nabla\Gamma(\cdot,\cdot,t_0,y)u(t_0,y), \nabla\Gamma(\cdot,\cdot,t_0,y)u(t_0,y)\rangle \,\mathrm{d}y \\ &= \left\langle A\nabla \int_{\mathbb{R}^N} \Gamma(\cdot,\cdot,t_0,y)u(t_0,y)\mathrm{d}y, \nabla \int_{\mathbb{R}^N} \Gamma(\cdot,\cdot,t_0,y)u(t_0,y)\mathrm{d}y \right\rangle \end{aligned}$$

the hypothesis. □

7.1.2 The Harnack inequality and $\mathcal{K}$-admissible curves

In this section we are going use the differential Harnack inequality to prove the Harnack inequality for Kolmogorov equations with constant coefficients. To do so, we introduce the notion of integral curves. Let $z_j = (t_j, x_j) \in (0,T) \times \mathbb{R}^N$, $j = 1,2$ with $t_1 < t_2$. An integral curve $\gamma = (\gamma_1, \dots, \gamma_N, t) \in C^\infty([t_1,t_2]; \mathbb{R}^{N+1})$ of the vector fields $\partial_{x_1}, \dots, \partial_{x_{m_0}}, -Y$ connecting z_1 and z_2 is called a $\mathcal{K}$-admissible curve. By integral curve we mean that the tangential vector $\dot\gamma$ of γ fulfills

$$\sum_{i=1}^{N} \dot\gamma_i(s)\partial_{x_i} + \dot\gamma_t(s)\partial_t = \dot\gamma(s) = \sum_{i=1}^{m_0} \dot\gamma_i(s)\partial_{x_i} - \langle\gamma(s), B\nabla\rangle + \partial_t = \sum_{i=1}^{m_0} \dot\gamma_i(s)\partial_{x_i} - \langle B^T\gamma(s), \nabla\rangle + \partial_t$$

for all $s \in [t_1,t_2]$. We recall that we write $\partial_{x_1}, \dots, \partial_{x_N}, \partial_t$ for the respective unit vectors in $\mathbb{R}^{N+1}$ and that $\langle \xi, \nabla\rangle = \sum_{i=1}^N \xi_i\partial_{x_i}$. The set of all $\mathcal{K}$-admissible curves connecting the points z_1 and z_2 is denoted by $\mathcal{A}_{z_1,z_2}$. Every $\mathcal{K}$-admissible path γ is a solution to the system

$$\dot\gamma^{(k)} = -B_k^T\gamma^{(k-1)} \tag{7.1.4}$$

for $k = 1, \dots, r$. Clearly, for the time component γ_t of γ it must hold $\gamma_t(s) = t_1 + (t_2 - t_1)s$ so that this component is uniquely determined by t_1 and t_2.

Proposition 7.1.4. For all $z_1, z_2 \in \mathbb{R}^{N+1}$ it holds that $\mathcal{A}_{z_1,z_2} \neq \emptyset$.

Proof. A proof of this statement can be found in [Cho40]. An essential ingredient in the proof is that the vector fields $\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y$ satisfy the Hörmander rank condition as shown in section 6.1. □

We define the cost of a $\mathcal{K}$-admissible curve γ connecting $z_1 = (t_1, x_1)$ and $z_2 = (t_2, x_2)$ as

$$\Phi(\gamma) = \int_{t_1}^{t_2} \langle A_0^{-1}\dot{\gamma}^{(0)}(s), \dot{\gamma}^{(0)}(s)\rangle \mathrm{d}s.$$

Clearly, this cost function is always nonnegative. After this preparation we can now state the Harnack inequality.

Theorem 7.1.5. *Let $T > 0$ and u be a positive solution of $\partial_t u = \mathcal{K}u$ in $(0,T) \times \mathbb{R}^N$. Given $0 < t_1 < t_2 < T$ and $x_1, x_2 \in \mathbb{R}^N$, it holds that*

$$u(t_1, x_1) \leq \left(\frac{t_2}{t_1}\right)^{\frac{Q}{2}} \exp\left(\frac{1}{4} \inf_{\gamma \in \mathcal{A}_{z_1,z_2}} \Phi(\gamma)\right) u(t_2, x_2).$$

Proof. The proof is based on the differential Harnack inequality presented in Section 7.1.1. We know that u satisfies

$$-Yu + \frac{Q}{2t}u \geq \frac{\langle A\nabla u, \nabla u\rangle}{u}$$

in $(0,T) \times \mathbb{R}^N$. We consider a smooth $\mathbb{R}^N$-valued vector field $W = W(t,x)$ and add the term $2\langle A\nabla u, W\rangle + u\langle AW, W\rangle$ to both sides of the above differential Harnack inequality to deduce

$$\begin{aligned} -Yu + \frac{Q}{2t}u + 2\langle A\nabla u, W\rangle + u\langle AW, W\rangle &\geq 2\langle A\nabla u, W\rangle + u\langle AW, W\rangle + \frac{\langle A\nabla u, \nabla u\rangle}{u} \\ &= \frac{1}{u}\langle A(\nabla u + uW), \nabla u + uW\rangle \geq 0. \end{aligned}$$

Choosing any $\gamma \in \mathcal{A}_{z_1,z_2}$, we define the smooth vector field W as

$$W(t,x) = \left(\frac{1}{2}A_0^{-1}\dot{\gamma}^{(0)}, 0, \dots, 0\right)$$

and calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}u(\gamma) = \left\langle (\nabla u, \partial_t u), \partial_t + \sum_{i=1}^{m_0} \dot{\gamma}_i \partial_{x_i} - \sum_{i=1}^{N} \gamma_i \sum_{j=1}^{N} b_{ij}\partial_{x_j} \right\rangle$$

$$
\begin{aligned}
&= \sum_{i=1}^{m_0}[\partial_{x_i}u](\gamma)\dot{\gamma}_i + \partial_t u + \left\langle (\nabla u, \partial_t u), -\sum_{i=1}^{N}\gamma_i \sum_{j=1}^{N} b_{ij}\partial_{x_j} \right\rangle \\
&= 2\left\langle A[\nabla u](\gamma), \frac{1}{2}A_0^{-1}\dot{\gamma}^{(0)} \right\rangle - \langle \gamma, B[\nabla u](\gamma)\rangle + \partial_t u(\gamma) \\
&= 2\left\langle A[\nabla u](\gamma), \frac{1}{2}A_0^{-1}\dot{\gamma}^{(0)} \right\rangle - [Yu](\gamma).
\end{aligned}
$$

The above equation together with the perturbed differential Harnack inequality shows that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}u(\gamma) + \frac{Q}{2t}u(\gamma) + \frac{1}{4}u(\gamma)\langle A_0^{-1}\dot{\gamma}^{(0)}, \dot{\gamma}^{(0)}\rangle \\
&= -[Yu](\gamma) + \frac{Q}{2t}u(\gamma) + 2\left\langle A[\nabla u](\gamma), \frac{1}{2}A_0^{-1}\dot{\gamma}^{(0)} \right\rangle + \frac{1}{4}u(\gamma)\langle A_0^{-1}\dot{\gamma}^{(0)}, \dot{\gamma}^{(0)}\rangle \geq 0.
\end{aligned}
$$

Dividing this inequality by u and then integrating with respect to the time variable of the $\mathcal{K}$-admissible path gives

$$
\ln(u(x_2,t_2)) - \ln(u(x_1,t_1)) + \frac{Q}{2}\left(\ln(t_2) - \ln(t_1)\right) + \frac{1}{4}\int_{t_1}^{t_2} \langle A_0^{-1}\dot{\gamma}^{(0)}(s), \dot{\gamma}^{(0)}(s)\rangle \mathrm{d}s \geq 0.
$$

Exponentiating and taking the infimum over all $\gamma \in \mathcal{A}_{z_1,z_2}$ shows the Harnack inequality. □

Remark 7.1.6. It remains to investigate the properties of $\inf \Phi$. We are going to show that the infimum is always attained by a polynomial $\mathcal{K}$-admissible path. In the case of $r = 1, 2$ one is able to calculate the exact value of $\inf \Phi$ and to show that the bound given in the Harnack inequality is actually sharp. We refer to [PP04a, Corollary 1.2] for further information on this matter. We are going to calculate $\inf \Phi$ for the classical Kolmogorov equation after some preparation.

Remark 7.1.7. In the celebrated paper [LY86] the authors prove a similar Harnack inequality for the heat equation on manifolds using a differential Harnack inequality. We want to highlight two differences between both approaches. The first is the first order term $\langle x, B\nabla\rangle$ which appears in the Kolmogorov equation. One might say that this leads to a different geometry of the space and therefore one has to consider the set of $\mathcal{K}$-admissible paths in the derivation of the Harnack inequality. Another difference is how the differential Harnack inequality is derived. In [LY86] the differential Harnack inequality is derived by making use of the strong maximum principle. The proof of the differential Harnack inequality for Kolmogorov type operators heavily relies on the fundamental solution. For the heat equation in $\mathbb{R}^n$ one could prove the differential Harnack inequality using the fundamental solution as well. To do so, denote by $g(t,x) = (4\pi t)^{-\frac{n}{2}}\exp\left(-\frac{|x|^2}{4t}\right)$ the fundamental solution of the heat equation and note that

$$\partial_{x_i} g(t,x) = -\frac{x_i}{2t} g(t,x) \tag{7.1.5}$$

as well as

$$\partial_{x_i}^2 g(t,x) = -\frac{g(t,x)}{2t} + \frac{x_i^2}{4t} g(t,x).$$

Therefore, we conclude

$$\partial_t g + \frac{n}{2t} g = \Delta g + \frac{n}{2t} g = \frac{n}{2t} g - \frac{n}{2t} g + \frac{|x|^2}{4t^2} g = \frac{|\nabla g|^2}{g}.$$

Arguing as in section 7.1.1, we can recover the classical differential Harnack inequality for positive solutions

$$\partial_t u + \frac{n}{2t} u \geq \frac{|\nabla u|^2}{u}.$$

Unfortunately, in general, there is no closed formula for the fundamental solution of the heat equation on a manifold.

Proposition 7.1.8. Given $z_1, z_2 \in \mathbb{R}^{N+1}$, we call a function $\gamma \in \mathcal{A}_{z_1,z_2}$ a critical point of Φ if

$$\mathrm{d}\Phi(\gamma, \varphi) := \int_{t_1}^{t_2} \langle A_0^{-1} \dot{\gamma}^{(0)}(s), \dot{\varphi}^{(0)}(s) \rangle \mathrm{d}s = 0$$

holds for all $\varphi \in \mathcal{A}_{0,0}$. A function $\gamma \in \mathcal{A}_{z_1,z_2}$ is a critical point of Φ if and only if γ is a minimum of Φ in $\mathcal{A}_{z_1,z_2}$.

Proof. Suppose that $\gamma \in \mathcal{A}_{z_1,z_2}$ is a critical point of Φ, then for every $\varphi \in \mathcal{A}_{z_1,z_2}$ it holds that $\gamma - \varphi \in \mathcal{A}_{0,0}$ and therefore we conclude

$$\Phi(\varphi) = \Phi(\gamma) + 2\mathrm{d}\Phi(\gamma, \gamma - \varphi) + \Phi(\gamma - \varphi) \geq \Phi(\gamma).$$

This shows that $\inf_{\varphi \in \mathcal{A}_{z_1,z_2}} \Phi(\varphi) = \Phi(\gamma)$. If conversely $\gamma \in \mathcal{A}_{z_1,z_2}$ is the minimum of Φ, then, given $\eta \in \mathcal{A}_{0,0}$, we have $\gamma + t\eta \in \mathcal{A}_{z_1,z_2}$. Therefore, it must hold that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \Phi(\gamma + t\eta) \Big|_{t=0} = 2\mathrm{d}\Phi(\gamma, \eta)$$

for all $\eta \in \mathcal{A}_{0,0}$ which means that γ is a critical point of Φ. □

Remark 7.1.9. In the remaining section we are going to consider only the case that $A_0 = \mathrm{Id}_{m_0}$. Since the definition of a $\mathcal{K}$-admissible path does not depend on A_0, we might just change the scalar product in the definition of Φ to the equivalent scalar product $\langle x, y \rangle_{A_0} := \langle A_0^{-1} x, y \rangle$ and therefore reduce the situation to the case $A_0 = \mathrm{Id}_{m_0}$.

We want to introduce a neat representation of paths γ which will be crucial in the proof of existence of critical points of Φ. Using this representation, we will deduce a simplified formula for $\mathrm{d}\Phi$. Setting $M_0 = \mathrm{Id}_{m_0}$ and $M_k = (-1)^k B_k^T \cdots B_1^T$ for $k = 1, \dots, r$, equation (7.1.4) implies

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\gamma^{(k)} = M_k \gamma^{(0)} \tag{7.1.6}$$

for $k = 0, \dots, r$. We introduce the linear subspaces $V_k \subset \mathbb{R}^{m_0}$ for $k = 0, \dots, r$ as follows. We define $V_0 = \mathcal{N}(M_1)$, $V_r = \mathcal{N}(M_r)^\perp$ and the remaining subspaces $V_k \subset \mathcal{N}(M_{k+1}) \setminus \mathcal{N}(M_k)$ inductively by the relation

$$V_k \oplus V_{k+1} \oplus \cdots \oplus V_r = \mathcal{N}(M_k)^\perp.$$

This is well-defined, since $\mathcal{N}(M_{k+1})^\perp \subset \mathcal{N}(M_k)^\perp$. Thus, every $\mathcal{K}$-admissible path γ can be uniquely represented as

$$\gamma^{(0)} = \gamma^{(0,0)} + \cdots + \gamma^{(0,r)}$$

for functions $\gamma^{(0,k)}$ with values in V_k. It is

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\gamma^{(k)} = M_k\gamma^{(0)} = M_k \sum_{j=0}^{r} \gamma^{(0,h)} = M_k \sum_{h=k}^{r} \gamma^{(0,h)} = M_k \gamma^{(0,k)} + M_k \sum_{h=k+1}^{r} \gamma^{(0,h)}$$

for $k = 0, \dots, r$, since $M_k\gamma^{(0,h)} = 0$ for all $h = 0, \dots, k-1$. Since each matrix $M_k \in \mathbb{R}^{m_k \times m_0}$ is of rank m_k, there exists a unique right inverse, denoted by M_k^{-1}. Given $k = 0, \dots, r$, we conclude that

$$\gamma^{(0,k)} = M_k^{-1} \frac{\mathrm{d}^k}{\mathrm{d}t^k}\gamma^{(k)} - \sum_{h=k+1}^{r} \gamma^{(0,h)}.$$

We have shown that for arbitrary $\gamma \in \mathcal{A}_{0,z}$ and $\eta \in \mathcal{A}_{0,0}$ we may rewrite $\mathrm{d}\Phi$ as

$$\begin{aligned} &\mathrm{d}\Phi(\gamma, \eta) \\ &= \sum_{k=0}^{r} \int_0^t \left\langle M_k^{-1} \frac{\mathrm{d}^{k+1}}{\mathrm{d}s^{k+1}}\gamma^{(k)}(s) - \sum_{h=k+1}^{r} \dot{\gamma}^{(0,h)}(s), M_k^{-1} \frac{\mathrm{d}^{k+1}}{\mathrm{d}s^{k+1}}\eta^{(k)}(s) - \sum_{h=k+1}^{r} \dot{\eta}^{(0,h)}(s) \right\rangle \mathrm{d}s. \end{aligned} \tag{7.1.7}$$

Under suitable conditions on $\eta^{(0)}$ we are able to deduce an even simpler representation of $\mathrm{d}\Phi$. This is the statement of the following lemma.

Lemma 7.1.10. Given $\gamma \in \mathcal{A}_{0,z}$ and $\eta \in \mathcal{A}_{0,0}$ such that $\eta^{(0)}(s) \in V_k$ for some $k = 0, \dots, r$

and for all $s \in [0,t]$, we get the following representation of $\mathrm{d}\Phi$

$$\mathrm{d}\Phi(\gamma,\eta) = (-1)^{k+1} \int_0^t \left\langle \frac{\mathrm{d}^{k+2}}{\mathrm{d}s^{k+2}} \gamma^{(0,k)}(s), M_k^{-1} \eta^{(k)}(s) \right\rangle \mathrm{d}s.$$

Proof. This is an immediate consequence of representation (7.1.7) and an orthogonality property of the spaces V_k. Since $\eta^{(0)} \in V_k$, it follows that $\eta^{(0,h)} = 0$ for all $h = 0, \dots, k-1, k+1, \dots, r$. The representation

$$\eta^{(0,h)} = M_h^{-1} \frac{\mathrm{d}^h}{\mathrm{d}t^h} \eta^{(h)} - \sum_{j=h+1}^{r} \eta^{(0,j)}$$

for all $h = 0, \dots, r$ shows $\eta^{(0,k)} = M_k^{-1} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \eta^{(k)}$. Using this information in equation (7.1.7) gives

$$\begin{aligned} \mathrm{d}\Phi(\gamma,\eta) &= \int_0^t \left\langle M_k^{-1} \frac{\mathrm{d}^{k+1}}{\mathrm{d}s^{k+1}} \gamma^{(k)}(s) - \sum_{h=k+1}^{r} \dot{\gamma}^{(0,h)}(s), M_k^{-1} \frac{\mathrm{d}^{k+1}}{\mathrm{d}s^{k+1}} \eta^{(k)}(s) \right\rangle \mathrm{d}s \\ &= \int_0^t \left\langle M_k^{-1} \frac{\mathrm{d}^{k+1}}{\mathrm{d}s^{k+1}} \gamma^{(k)}(s), M_k^{-1} \frac{\mathrm{d}^{k+1}}{\mathrm{d}s^{k+1}} \eta^{(k)}(s) \right\rangle \mathrm{d}s, \end{aligned}$$

since $\eta^{(0,k)}(s) \in V_k$ for all $s \in [0,t]$ and hence $M_k^{-1} \frac{\mathrm{d}^{k+1}}{\mathrm{d}s^{k+1}} \eta^{(k)}(s) = \frac{\mathrm{d}}{\mathrm{d}t} \eta^{(0,k)}(s) \in V_k$ for all $s \in [0,t]$. Integrating by parts and taking into account that due to $\eta \in \mathcal{A}_{0,0}$ there is no contribution at the boundary, we conclude

$$\begin{aligned} \mathrm{d}\Phi(\gamma,\eta) &= \int_0^t \left\langle M_k^{-1} \frac{\mathrm{d}^{k+1}}{\mathrm{d}s^{k+1}} \gamma^{(k)}(s), M_k^{-1} \frac{\mathrm{d}^{k+1}}{\mathrm{d}s^{k+1}} \eta^{(k)}(s) \right\rangle \mathrm{d}s \\ &= (-1)^{k+1} \int_0^t \left\langle M_k^{-1} \frac{\mathrm{d}^{2k+2}}{\mathrm{d}s^{2k+2}} \gamma^{(k)}(s), M_k^{-1} \eta^{(k)}(s) \right\rangle \mathrm{d}s \\ &= (-1)^{k+1} \int_0^t \left\langle M_k^{-1} \frac{\mathrm{d}^{k+2}}{\mathrm{d}s^{k+2}} M_k \sum_{h=k}^{r} \gamma^{(0,h)}, M_k^{-1} \eta^{(k)}(s) \right\rangle \mathrm{d}s \\ &= (-1)^{k+1} \int_0^t \left\langle \frac{\mathrm{d}^{k+2}}{\mathrm{d}s^{k+2}} \sum_{h=k}^{r} \gamma^{(0,h)}(s), M_k^{-1} \eta^{(k)}(s) \right\rangle \mathrm{d}s. \end{aligned}$$

Noting that $M_k^{-1} \eta^{(k)} \in V_k$, we deduce the hypothesis by orthogonality of the appearing terms. □

By choosing a suitable test function in lemma 7.1.10 we are going to deduce that the minimizers of Φ are polynomial. To simplify the proof, we introduce yet another family of subspaces. Given $v \in V_k$, by setting $\tilde{v} = (v, 0, \dots, 0) \in \mathbb{R}^N$, we may interpret V_k as a linear subspace of

$\mathbb{R}^N$. We introduce the linear subspaces $W_0, \dots, W_r$ of $\mathbb{R}^N$ defined by

$$W_k = V_k \oplus B^T V_k \oplus \cdots \oplus (B^T)^k V_k$$

for $k = 0, \dots, r$. The direct sum of all V_k is $\mathbb{R}^{m_0}$ so, since every B_j^T is of rank m_j, the direct sum of all W_k is $\mathbb{R}^N$. Therefore, every γ might be uniquely written as

$$\gamma = \sum_{k=0}^{r} \sum_{j=0}^{k} \gamma^{(j,k)}$$

for functions $\gamma^{(j,k)}$ with values in $(B^T)^j V_k$. The structure of these subspaces can be seen in the following chart:

$$
\begin{array}{ccccccccccccc}
 & & \mathbb{R}^{m_0} & & V_0 & \oplus & V_1 & \oplus & V_2 & \oplus & \cdots & \oplus & V_r \\
 & & \mathbb{R}^{m_1} & & & & B^T V_1 & \oplus & B^T V_2 & \oplus & \cdots & \oplus & B^T V_r \\
 & & \mathbb{R}^{m_2} & & & & & & (B^T)^2 V_2 & \oplus & \cdots & \oplus & (B^T)^2 V_r \\
\mathbb{R}^N & = & \vdots & = & & & & & & \ddots & \ddots & \ddots & \vdots \\
 & & \vdots & & & & & & & & \ddots & \ddots & \vdots \\
 & & \vdots & & & & & & & & & \ddots & \vdots \\
 & & \mathbb{R}^{m_k} & & & & & & & & & & (B^T)^r V_r.
\end{array}
$$

The columns represent the spaces $W_0, \dots, W_r$, while one also sees which subspaces determine the k-th component in $\mathbb{R}^N$. We recall that due to the fact that B is a block matrix the subspace $(B^T)^j V_k$ only adds to the j-th component of a vector in $\mathbb{R}^N$. To be more precise, it holds $\gamma^{(j,k)} \in \left\{ x \in \mathbb{R}^N \mid x^{(i)} = 0 \ \forall i \neq j \right\}$. Therefore, we might interpret $\gamma^{(j,k)}$ either as a vector in $\mathbb{R}^N$ or in $\mathbb{R}^{m_k}$.

With this convention in mind it holds

$$\dot{\gamma}^{(k,j)} = -B_k^T \gamma^{(k-1,j)} \tag{7.1.8}$$

for all $k = 0, \dots, r$ and any $j = k, \dots, r$. This can be seen by differentiating the representation of the k-th component of γ in terms of W_k

$$\gamma^{(k)} = \sum_{j=k}^{r} \gamma^{(k,j)}.$$

Using equation (7.1.4), we get that

$$\sum_{j=k}^{r} \dot{\gamma}^{(k,j)} = \dot{\gamma}^{(k)} = -B_k^T \left(\sum_{j=k-1}^{r} \gamma^{(k-1,j)} \right) = -\sum_{j=k}^{r} B_k^T \gamma^{(k-1,j)},$$

since $V_{k-1} \subset \mathcal{N}(M_k)$. By the uniqueness of the representation in terms of W_k, we deduce equation (7.1.8).

Equation (7.1.8) readily implies

$$\frac{\mathrm{d}^h}{\mathrm{d}s^h} \gamma^{(k,j)} = (-1)^h M_k \cdots M_{k-h+1} \gamma^{(k-h,j)} = M_k M_{k-h}^{-1} \gamma^{(k-h,j)} \tag{7.1.9}$$

for all $k = 0, \ldots, r$, $h = 0, \ldots, k$ and $j = k, \ldots, r$.

Proposition 7.1.11. Let $\gamma \in \mathcal{A}_{0,z}$. Then γ is a critical point of Φ in $\mathcal{A}_{0,z}$ if and only if for all $k = 0, \ldots, r$ the function $\gamma^{(0,k)}$ is a polynomial of degree less or equal $k+1$.

Proof. To deduce the necessity of this condition, let $\gamma \in \mathcal{A}_{0,z}$ be a critical point. Given $k \in \{0, \ldots, r\}$, we choose $v \in V_k$ and $\varphi \in C_0^\infty([0,t];\mathbb{R}) = \{\alpha \in C^\infty([0,t];\mathbb{R}) \mid \alpha(0) = \alpha(t) = 0\}$. In lemma 7.1.10 we want to use the test function defined by

$$\eta = \left(\frac{\mathrm{d}^k \varphi}{\mathrm{d}s^k} v, \frac{\mathrm{d}^{k-1}\varphi}{\mathrm{d}s^{k-1}} M_1 v, \ldots, \varphi M_k v, 0, \ldots 0 \right).$$

Clearly, it is $\eta \in C^\infty([0,t];\mathbb{R}^{N+1})$. Since it is $\varphi(0) = \varphi(t) = 0$, we have $\eta(0) = \eta(t) = 0$. Further, we have $\eta^{(0)}(s) \in V_k$ for all $s \in [0,t]$ because φ is scalar and $v \in V_k$ and it holds that

$$\dot{\eta}^{(j)} = \frac{\mathrm{d}}{\mathrm{d}s} \frac{\mathrm{d}^{k-j}}{\mathrm{d}s^{k-j}} \varphi M_j v = -B_j^T M_{j-1} \frac{\mathrm{d}^{k-j+1}}{\mathrm{d}s^{k-j+1}} = -B_j^T \eta^{(j-1)}$$

for all $j = 1, \ldots, k$. Finally, it holds

$$\dot{\eta}^{(k+1)} = 0 = -B_{k+1}^T \varphi M_k v = -B_{k+1}^T \eta^{(k)},$$

since $v \in V_k \subset \mathcal{N}(M_{k+1})$ and for $j = k+2, \ldots, r$ above identity trivially holds, too. Therefore, we conclude $\eta \in \mathcal{A}_{0,0}$ and are able to apply lemma 7.1.10 to deduce

$$\begin{aligned} 0 = \mathrm{d}\Phi(\gamma, \eta) &= (-1)^{k+1} \int_0^t \left\langle \frac{\mathrm{d}^{k+2}}{\mathrm{d}s^{k+2}} \gamma^{(0,k)}(s), M_k^{-1} \eta^{(k)} \right\rangle \mathrm{d}s \\ &= (-1)^{k+1} \int_0^t \left\langle \frac{\mathrm{d}^{k+2}}{\mathrm{d}s^{k+2}} \gamma^{(0,k)}(s), M_k^{-1} M_k \varphi v \right\rangle \mathrm{d}s \\ &= (-1)^{k+1} \int_0^t \left\langle \frac{\mathrm{d}^{k+2}}{\mathrm{d}s^{k+2}} \gamma^{(0,k)}(s), \varphi v \right\rangle \mathrm{d}s \end{aligned}$$

by taking into account that $v \in V_k \subset \mathcal{N}(M_k)^\perp$ and thus $M_k^{-1}M_k v = v$. Since it is $\gamma^{(0,k)}(s) \in V_k$ for all s and $v \in V_k$ was arbitrary, we conclude using the fundamental lemma of calculus of variations that $\frac{\mathrm{d}^{k+2}}{\mathrm{d}s^{k+2}}\gamma^{(0,k)} = 0$. This shows that for all $k = 0, \dots, r$ the function $\gamma^{(0,k)}$ is a polynomial of degree less or equal than $k+1$.

To show the converse implication, we observe that by above calculation we get $\mathrm{d}\Phi(\gamma, \eta) = 0$ for all $\eta \in \mathcal{A}_{0,0}$ such that η is of above form. The claim would now follow by linearity of $\mathrm{d}\Phi$ if we may write every $\eta \in \mathcal{A}_{0,0}$ as a linear combination of such functions. To see this, we need the representation of η in terms of the subspaces W_k. Surely, we can write $\eta = \sum_{k=0}^r \eta_k$ for functions η_k with values in W_k. Let $k = 0, \dots, r$ and assume that $V_k = \operatorname{span}\{v\}$, then there are functions $\eta_k^i \in C^\infty([0,t];\mathbb{R})$, $i = 0, \dots, k$ such that

$$\eta_k = \left(\eta_k^k v, \eta_k^{k-1} M_1 v, \eta_k^{k-2} M_2 v, \dots, \eta_k^0 M_k v, 0 \dots, 0\right).$$

Differentiating the k-th component j-times and applying equation (7.1.4), we see that

$$\begin{aligned} \frac{\mathrm{d}^j}{\mathrm{d}s^j}\eta_k^0 M_k v = \frac{\mathrm{d}^j}{\mathrm{d}s^j}\eta_k^{(k)} &= (-1)^j B_k^T \cdots B_{k-j+1}^T \eta_k^{(k-j)} v \\ &= \eta_k^j (-1)^j B_k^T \cdots B_{k-j+1}^T M_{k-j} v = \eta_k^j M_k v \end{aligned}$$

and therefore conclude $\frac{\mathrm{d}^j}{\mathrm{d}s^j}\eta_k^0 = \eta_k^j$ for all $j = 0, \dots, k$. So η_k is of the desired shape. Also due to the the direct sum property it must hold that $\eta_i^k \in C_0^\infty([0,t];\mathbb{R})$. If $\dim V_k > 1$, we proceed as before but choose a base of V_k and therefore have to repeat the above step for every basis vector. This shows that above condition is actually sufficient for γ to be a critical point of $\mathrm{d}\Phi$. $\square$

Example 7.1.12. We want to calculate the admissible paths corresponding to the Kolmogorov operator which are critical points of Φ. Let $t > 0$ and $x \in \mathbb{R}^N$. An admissible path to the points 0 and (t,x) is a function $\gamma \in C^\infty([0,t];\mathbb{R}^{N+1})$ such that $\gamma(0) = 0$, $\gamma(t) = (t,x)$ and

$$\dot{\gamma}^{(1)}(s) = \gamma^{(0)}(s) \tag{7.1.10}$$

or all $s \in [0,t]$. The subspaces V_0, V_1 are given by $V_0 = \{0\}$ and $V_1 = \mathbb{R}^n$. Therefore, by proposition 7.1.11, we know that $\gamma^{(0)}$ is a polynomial of degree at most 2. We make the ansatz $\gamma^{(0)}(s) = \alpha_1 s^2 + \alpha_2 s + \alpha_3$ with vectors $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}^n$. In view of equation (7.1.10) it must hold that $\gamma^{(1)}(s) = \frac{1}{3}s^3\alpha_1 + \frac{1}{2}\alpha_2 s^2 + \alpha_3 s + \alpha_4$ for a vector $\alpha_4 \in \mathbb{R}^n$. The conditions above then show that it must hold

$$\alpha_3 = 0,\ \alpha_1 t^2 + \alpha_2 t = x^{(0)},\ \alpha_4 = 0 \text{ and } \frac{1}{3}t^3\alpha_1 + \frac{1}{2}\alpha_2 t^2 = x^{(1)}. \tag{7.1.11}$$

This linear system (7.1.11) can be solved for α_1, α_2. It is

$$\alpha_1 = \frac{3}{t^2} x^{(0)} - \frac{6}{t^3} x^{(1)} \text{ and } \alpha_2 = -\frac{2}{t} x^{(0)} + \frac{6}{t^2} x^{(1)}$$

and therefore

$$\gamma^{(0)}(s) = \left(\frac{3s^2}{t^2} - \frac{2s}{t} \right) x^{(0)} + \left(-\frac{6s^2}{t^3} + \frac{6s}{t^2} \right) x^{(1)}.$$

We calculate

$$\dot{\gamma}^{(0)}(s) = \left(\frac{6s}{t^2} - \frac{2}{t} \right) x^{(0)} + \left(-\frac{12s}{t^3} + \frac{6}{t^2} \right) x^{(1)}$$

and deduce

$$\begin{aligned}
\left\langle \dot{\gamma}^{(0)}(s), \dot{\gamma}^{(0)}(s) \right\rangle &= \left(\frac{6s}{t^2} - \frac{2}{t} \right)^2 \|x^{(0)}\|^2 + 2 \left(\frac{6s}{t^2} - \frac{2}{t} \right) \left(-\frac{12s}{t^3} + \frac{6}{t^2} \right) \left\langle x^{(0)}, x^{(1)} \right\rangle \\
&\quad + \left(-\frac{12s}{t^3} + \frac{6}{t^2} \right)^2 \|x^{(1)}\|^2 \\
&= \left(\frac{36s^2}{t^4} - \frac{24s}{t^3} + \frac{4}{t^2} \right)^2 \|x^{(0)}\|^2 \\
&\quad + \left(-\frac{144s^2}{t^5} + \frac{72s}{t^4} + \frac{48s}{t^4} - \frac{24}{t^3} \right) \left\langle x^{(0)}, x^{(1)} \right\rangle \\
&\quad + \left(\frac{144s^2}{t^6} - \frac{144s}{t^5} + \frac{36}{t^4} \right)^2 \|x^{(1)}\|^2.
\end{aligned}$$

Integrating over the interval $[0, t]$, we conclude

$$\Phi(\gamma) = \int_0^t \left\langle \dot{\gamma}^{(0)}(s), \dot{\gamma}^{(0)}(s) \right\rangle \mathrm{d}s = \frac{4}{t} \|x^{(0)}\|^2 - \frac{12}{t^2} \left\langle x^{(0)}, x^{(1)} \right\rangle + \frac{12}{t^3} \|x^{(1)}\|^2.$$

We recall that in example 3.1.13 we calculated

$$\langle \mathcal{C}(t)^{-1} x, x \rangle = \frac{4}{t} \|x^{(0)}\|^2 - \frac{12}{t^2} \left\langle x^{(0)}, x^{(1)} \right\rangle + \frac{12}{t^3} \|x^{(1)}\|^2$$

so that it is $\Phi(\gamma) = \langle \mathcal{C}(t)^{-1} x, x \rangle$ and by proposition 7.1.11, it holds $\inf_{\varphi \in \mathcal{A}_{0,(t,x)}} \Phi(\varphi) = \langle \mathcal{C}(t)^{-1} x, x \rangle$ for all $(t, x) \in \mathbb{R}^{N+1}$.

Theorem 7.1.13. *For all $z_1, z_2 \in \mathbb{R}^{N+1}$ there exists a unique $\gamma \in \mathcal{A}_{z_1, z_2}$ such that*

$$\Phi(\gamma) = \inf_{\varphi \in \mathcal{A}_{z_1, z_2}} \Phi(\varphi).$$

Proof. We are going to show that it suffices to prove the claim for $z_1 = 0$ and $z_2 = z$. We recall that $\mathcal{K} - \partial_t$ is invariant with respect to the left translation defined by

$$(t,x) \circ (\xi,\tau) = (\xi + E(\tau)x, t+\tau)$$

for all $(t,x),(\xi,\tau) \in \mathbb{R}^{N+1}$. Let us assume that the hypothesis holds true for $z_1 = 0$ and $z_2 = z$. Given arbitrary $z_1 = (t_1, x_1)$ and $z_2 = (t_2, x_2)$, we define $\tilde{z}_1 = 0$ and

$$\tilde{z}_2 = (t_2 - t_1, x_2 - E(t_2-t_1)x_1).$$

Let $z_1 = (x_1, t_1), z_2 = (x_2, t_2) \in \mathbb{R}^N$ and define $z = (x_2 - E(t_2-t_1), t_2 - t_1)$. We want to show that there is a one to one correspondence of $\mathcal{A}_{z_1,z_2}$ and $\mathcal{A}_{0,z}$. Let $\gamma \in \mathcal{A}_{0,z}$ and define $\tilde{\gamma}(t) = \gamma(t-t_1) + E(t-t_1)x_1$ for $t \in [t_1,t_2]$. Then $\tilde{\gamma} \in C^\infty([t_1,t_2];\mathbb{R}^{N+1})$ and $\tilde{\gamma}(t_1) = \gamma(0) + E(0)x_1 = x_1$ as well as $\tilde{\gamma}(t_2) = x_2 - E(t_2-t_1)x_1 + E(t_2-t_1)x_1 = x_2$. It remains to verify the condition on the tangential vector. It holds

$$\begin{aligned}
\dot{\tilde{\gamma}}(t) &= \dot{\gamma}(t-t_1) + \langle E(t_2-t_1)x_1, B\nabla\rangle \\
&= \sum_{i=1}^{m_0} \dot{\gamma}_i(t-t_1) - \langle \gamma(t-t_1), B\nabla\rangle + \partial_t + \langle E(t_2-t_1)x_1, B\nabla\rangle \\
&= \sum_{i=1}^{m_0} \dot{\tilde{\gamma}}_i(t) - \langle \tilde{\gamma}(t), B\nabla\rangle + \partial_t \\
&= \sum_{i=1}^{m_0} \dot{\tilde{\gamma}}_i(t) - Y(\tilde{\gamma})
\end{aligned}$$

so that $\tilde{\gamma} \in \mathcal{A}_{z_1,z_2}$. Conversely, given $\gamma \in \mathcal{A}_{z_1,z_2}$, we define $\tilde{\gamma}(t) = \gamma(t+t_1) - E(t)x_1$, then it holds $\tilde{\gamma} \in \mathcal{A}_{0,z}$. This transformation is clearly injective and the cost function is actually invariant under this transformation. To be more precise, let $\gamma \in \mathcal{A}_{0,z}$ and define $\tilde{\gamma} \in \mathcal{A}_{z_1,z_2}$ as above, then

$$\begin{aligned}
\Phi(\gamma) &= \int_0^t \langle A_0^{-1}\dot{\gamma}^{(0)}(s), \dot{\gamma}^{(0)}(s)\rangle \mathrm{d}s \\
&= \int_{t_1}^{t_2} \langle A_0^{-1}\dot{\tilde{\gamma}}^{(0)}(r), \dot{\tilde{\gamma}}^{(0)}(r)\rangle \mathrm{d}r + \int_{t_1}^{t_2} \langle A_0^{-1}[B^T E(t-t_1)x_1]^{(0)}, [B^T E(t-t_1)x_1]^{(0)}\rangle \mathrm{d}r \\
&= \int_{t_1}^{t_2} \langle A_0^{-1}\dot{\tilde{\gamma}}^{(0)}(r), \dot{\tilde{\gamma}}^{(0)}(r)\rangle \mathrm{d}r + 0 = \Phi(\tilde{\gamma}),
\end{aligned}$$

since $[B^T x]_i = 0$ for all $x \in \mathbb{R}^N$ and all $i = 1,\dots,m_0$. We have shown that it suffices to prove the hypothesis in the case of $z_1 = 0$ and $z_2 = z \in \mathbb{R}^{N+1}$. Let $z \in \mathbb{R}^{N+1}$ and $\gamma \in \mathcal{A}_{0,z}$. It holds

$$\frac{\mathrm{d}^h}{\mathrm{d}s^h}\gamma^{(k,k)} = M_k M_{k-h}^{-1}\gamma^{(k-h,k)}$$

for all $h = 0, \dots, k$. From lemma 7.1.11 we know that

$$\gamma^{(k,k)}(s) = \sum_{j=0}^{2k+1} \alpha_j s^j$$

with vectors $\alpha_j \in \mathbb{R}^{m_k - m_{k+1}}$. It must hold $\gamma(0) = 0$ and therefore it is

$$0 = \gamma^{(k,k)} = \dot{\gamma}^{(k,k)}(0) = \cdots = \frac{\mathrm{d}^k}{\mathrm{d}s^k}\gamma^{(k,k)}(0)$$

or equivalently $\alpha_0 = \cdots = \alpha_k = 0$. Consequently, $\gamma^{(k,k)} = s^{k+1}g(s)$ for a polynomial

$$g(s) = \sum_{j=0}^{k} \frac{\beta_j}{j!}(s-t)^j$$

with vectors $\beta_j \in \mathbb{R}^{m_k - m_{k+1}}$. The terminal condition $\gamma(t) = x$ leads to

$$\begin{aligned} x^{(k,k)} &= \gamma^{(k,k)}(t) = t^{k+1}\beta_0 \\ -B_k^T x^{(k-1,k)} &= \dot{\gamma}^{(k,k)}(t) = (k+1)t^k\beta_0 + t^k\beta_1 \\ &\vdots \\ (-1)^k M_k x^{(0,k)} &= \frac{\mathrm{d}^k}{\mathrm{d}s^k}\gamma^{(k,k)}(t) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k+1)!}{(j+1)!} t^{j+1}\beta_j. \end{aligned}$$

Thus, defining the vectors β_j inductively from top to bottom leads to a unique solution. We have shown that there is a unique polynomial $\mathcal{K}$-admissible path satisfying the sufficient condition in lemma 7.1.11. This shows the theorem. □

Example 7.1.14. Let us consider the Kolmogorov equation. In example 7.1.12 we have calculated $\Phi(\gamma)$ for the $\mathcal{K}$-admissible minimizer $\gamma \in \mathcal{A}_{0,z}$. Arguing as in the proof of theorem 7.1.13, we can also deduce the value of $\Phi(\gamma)$, $\gamma \in \mathcal{A}_{z_1,z_2}$ for arbitrary $z_1, z_2 \in \mathbb{R}^{N+1}$. It holds

$$\Phi(\gamma) = \left\langle \mathcal{C}^{-1}(t_2 - t_1)(x_2 - E(t_2 - t_1)x_1), x_2 - E(t_2 - t_1)x_1 \right\rangle$$

for the minimizer $\gamma \in \mathcal{A}_{z_1,z_2}$ of Φ in $\mathcal{A}_{z_1,z_2}$.

7.2 Kolmogorov equations with rough coefficients

Inspired by the work of Ennio De Giorgi, Jürgen Moser and John Nash it is natural to ask whether one can prove a Harnack inequality for the Kolmogorov equation with rough, i.e. bounded and measurable, diffusion coefficients, too. In this sense, let $n \in \mathbb{N}$, $N = 2n$, $T > 0$ and $A \in L^\infty(\mathbb{R}^{N+1}; \mathbb{R}^{n\times n})$ be a bounded and measurable strictly elliptic function of matrices. In particular, there exists a constant $\lambda > 0$ such that

$$\lambda \left|\xi\right|^2 \leq \langle A(t,v,x)\xi, \xi\rangle \leq \frac{1}{\lambda}\left|\xi^2\right|$$

for all $\xi \in \mathbb{R}^n$ and all $(t,v,x) \in \mathbb{R}^{N+1}$. We consider the partial differential equation

$$\partial_t u(t,v,x) + v \cdot \nabla_x u(t,v,x) = \operatorname{div}_v(A(t,v,x)\nabla u(t,v,x)).$$

The answer to the question whether a Harnack inequality holds in this case has been found recently by the authors of [GIMV16]. We are going to present their main result in the following section. To simplify the statement, we introduce the cylinders

$$Q_r(t,v,x) = (t-r^2, t] \times B_r(v) \times B_{r^3}(x) \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$$

for $r > 0$. We have already seen that the scaling of these cylinders is very natural in the context of the Kolmogorov equation. This is due to the fact that the dilation group preserves the structure of the cylinders. We want to define the notion of a weak solution of equation (7.2). The subscript t, v, x for example in L^p_x or $W^{p,m}_v$ is to clarify that a function is an element of these spaces only in the specified variable.

Definition 7.2.1. Let I be a bounded interval, $\Omega_v \subset \mathbb{R}^n$ and $\Omega_x \subset \mathbb{R}^n$ open sets. We say that a function $u \in L^\infty_t(I; L^2_{v,x}(\Omega_v \times \Omega_x)) \cap L^2_{t,x}(I \times \Omega_x; W^{1,2}_v(\Omega_v))$ such that $\partial_t u + v \cdot \nabla_x u \in L^2_{t,x}(I \times \Omega_x; W^{-1,2}_v(\Omega_v))$, satisfying

$$\int_{I\times\Omega_v\times\Omega_x} (\partial_t\varphi + v\cdot\nabla_x\varphi)u\,\mathrm{d}(t,v,x) = (\leq) \int_{I\times\Omega_v\times\Omega_x} \langle A(t,v,x)\nabla u, \nabla\varphi\rangle \mathrm{d}(t,v,x)$$

for all $\varphi \in C^\infty_c(I \times \Omega_v \times \Omega_x)$, is a weak (super-)solution of equation (7.2).

Theorem 7.2.2. *Let u be a nonnegative weak solution of equation* (7.2) *in the cylinder $Q_1(0,0,0)$. There are constants $C > 1$ and $R, \delta \in (0,1)$ only depending on dimension and the ellipticity constant of A such that*

$$\sup_{Q_R(0,0,-\delta)} u \leq C \inf_{Q_R(0,0,0)} u.$$

While this Harnack inequality is only stated for cylinders close to the origin one can transform this result to arbitrary points in space. This calculation and a geometric interpretation is written down in the article [AEP18].

A difference to the classical elliptic-parabolic case is that the cylinders in the statement of the Harnack inequality cannot be chosen arbitrarily. The admissible location of such cylinders is deeply connected to the notion of admissible curves which we have studied in section 7.1.2. More information on this topic can be found in [AEP18].

The local boundedness of solutions to the Kolmogorv equation with rough diffusion coefficients using the Moser's iterative method has been studied first in the article [PP04b]. A different proof is presented in [GIMV16].

The proof of the Harnack inequality in [GIMV16] uses the ideas of Ennio De Girogi. The Harnack inequality in the parabolic case was independently proven by Jürgen Moser in [Mos64]. One important tool in the proof of Jürgen Moser is the weak Harnack inequality. It is still open whether a weak Harnack inequality holds in the kinetic case, too. Let us draft a possible version of a weak Harnack inequality for the equation (7.2).

Question 7.2.3. Let u be a nonnegative supersolution of equation (7.2) in $Q_1 = (0,1) \times B_1(0) \times B_1(0)$. Is there a $p \in (0,\infty)$ such that there exists a constant $C = C(n,\lambda,p)$, a positive radius $r_0 < 1$ and times $t_0 < t_1 \in (0,1)$ so that it holds

$$\left(\frac{1}{|Q_-|}\int_{Q_-} u^p \mathrm{d}(t,v,x)\right)^{\frac{1}{p}} \leq C \inf_{Q_+} u \tag{7.2.1}$$

for the nonoverlapping cylinders

$$\begin{aligned} Q_- &= [0,t_0] \times B_r(0) \times B_{r^3}(0) \\ Q_- &= [t_1,1] \times B_r(0) \times B_{r^3}(0). \end{aligned}$$

Remark 7.2.4. (i) We recall that in the kinetic case the sets in the Harnack inequality cannot be chosen arbitrarily. Thus, in contrast to the classical statement we should adapt question 7.2.3 to ask only for the existence of such cylinders so that the inequality holds.

(ii) In the article [IS16] the authors derive a weak Harnack inequality for one $p > 0$ for the Boltzmann equation without cutoff. It is the consequence of a result on the propagation of the minima of supersolutions and a so-called ink-spots theorem. A similar result on the propagation of the minima of supersolutions holds also in the case of the Kolmogorov equation. It seems likely that one can adapt these methods to our case to prove a weak Harnack inequality at least for one p.

We are going to give an upper bound on the optimal p such that a weak Harnack inequality as in question 7.2.3 can hold. We are going to use the truncated fundamental solution of the Kolmogorov equation to show that it must hold $p < 1 + \frac{1}{2n} = 1 + \frac{2}{Q}$. Let us highlight the fact that in the classical situation the optimal exponent is given by $1 + \frac{2}{N}$ which corresponds to homogenous dimension equal to $Q = N$, i.e. the diffusive equation without drift.

Proposition 7.2.5. If $p \geq 1 + \frac{1}{2n}$, then a weak Harnack inequality as in question 7.2.3 cannot hold.

Proof. We consider the case $A(t, v, x) = \mathrm{Id}_n$. We have already studied this equation intensively in the preceding chapters. Let us denote by Γ its fundamental solution as defined in section 3.1.1. Let $k \in \mathbb{N}$. We introduce the truncated fundamental solution as

$$\Phi_k(t, v, x) = \begin{cases} \Gamma(t, v, x) & t \geq \frac{1}{k} \\ \Gamma\left(\frac{1}{k}, v, x\right) & t \leq \frac{1}{k}. \end{cases}$$

This is a weak supersolution to the Kolmogorov equation in a suitable cylinder. We note that for $t \geq \frac{1}{k}$ it is indeed a solution. Moreover, one can show that Φ_k is a weak supersolution in some set $Q = (0, 1) \times B_R(0) \times B_R(0)$ where the radius R depends on the dimension. However, the necessary calculation to obtain this result is not that interesting. For this reason it is postponed to proposition B.0.8 in the appendix.

Let us estimate the infimum of Φ_k on any cylinder $Q_+ \subset Q$. It holds

$$\frac{3}{t^2}\langle v, x\rangle \leq \frac{3}{2\varepsilon t^3}|x|^2 + \frac{3\varepsilon}{2t}|v|$$

for all $\varepsilon \in (0, 1)$ so that if $\varepsilon \in (\frac{1}{2}, \frac{2}{3})$, we deduce that the exponent in the fundamental solution is nonpositive and therefore it is $\Phi_k(t, v, x) \leq \frac{c_0}{t^{2n}}$ for all $t \geq \frac{1}{k}$. Let $0 < t_0 < t_1 < 1$, then for k large enough it holds

$$\inf_{(t_1,1)\times B_R(0)\times B_R(0)} \Phi_k \leq \frac{c_0}{t_1^{2n}}. \tag{7.2.2}$$

We are interested in the L^p-integral of Φ_k on an cylinder $Q = (0, t_0) \times B_r(0) \times B_{r'}(0)$, where $0 < r, r' < R$. Using the Cauchy-Schwarz inequality and Young's inequality we estimate

$$\begin{aligned}
&\int_{\frac{1}{k}}^{t_0}\int_{B_r(0)}\int_{B_{r'}(0)} \Phi_k(t, v, x)^p \mathrm{d}(t, v, x) \\
&= c_0 \int_{\frac{1}{k}}^{t_0} \frac{1}{t^{2np}} \int_{B_r(0)} \exp\left(-\frac{p}{t}|v|^2\right) \int_{B_{r'}(0)} \exp\left(\frac{3p}{t^2}\langle x, v\rangle - \frac{3p}{t^3}|x|^2\right) \mathrm{d}x\mathrm{d}v\mathrm{d}t \\
&\geq c_0 \int_{\frac{1}{k}}^{t_0} \frac{1}{t^{2np}} \int_{B_r(0)} \exp\left(-\frac{p}{t}|v|^2\right) \int_{B_{r'}(0)} \exp\left(-\frac{3p}{2t^3}|x|^2 - \frac{3p}{2t}|v|^2 - \frac{3p}{t^3}|x|^2\right) \mathrm{d}x\mathrm{d}v\mathrm{d}t.
\end{aligned}$$

Further, we estimate

$$\begin{aligned}\int_{B_{r'}(0)} \exp\left(-\frac{9p}{2t^3}\left|x\right|^2\right) \mathrm{d}x &= c_1 \int_0^{r'} \exp\left(-\frac{9p}{2t^3}\left|z\right|^2\right) z^{n-1} \mathrm{d}z \\ &= c_1 t^{\frac{3}{2}n} \int_0^{r't^{-\frac{3}{2}}} \exp\left(-\frac{9p}{2}\left|y\right|^2\right) y^{n-1} \mathrm{d}y \\ &\geq c_1 t^{\frac{3}{2}n} \int_0^{r'} \exp\left(-\frac{9p}{2}\left|y\right|^2\right) y^{n-1} \mathrm{d}y = c_2 t^{\frac{3}{2}n}\end{aligned}$$

and similarly

$$\int_{B_r(0)} \exp\left(-\frac{5p}{2t}\left|v\right|^2\right) \mathrm{d}v \geq c_3 t^{\frac{1}{2}n}$$

where the positive constants c_1, c_2, c_3 only depend on dimension n and p. Combining the latter estimates shows that

$$\int_{\frac{1}{k}}^{t_0} \int_{B_r(0)} \int_{B_{r'}(0)} \Phi_k(t,v,x)^p \mathrm{d}(t,v,x) \geq c_4 \int_{\frac{1}{k}}^{t_0} t^{-2np+2n} \mathrm{d}t$$

for some constant $c_4 = c_4(d,p) > 0$. Finally it holds

$$\int_0^{\frac{1}{k}} \int_{B_r(0)} \int_{B_{r'}(0)} \Phi_k(t,v,x)^p \mathrm{d}(t,v,x) \geq 0.$$

All in all, we conclude

$$\|\Phi_k\|_{p,Q}^p \geq c_5 \int_{\frac{1}{k}}^{t_0} t^{-2np+2n} \mathrm{d}t.$$

The latter integral diverges if and only if $-2np + 2n \leq -1$, equivalently $p \geq 1 + \frac{2}{4n}$. This shows, together with equation 7.2.2, that the weak Harnack inequality cannot hold in the case $p \geq 1 + \frac{1}{2n}$. □

Let us comment on some peculiarities when trying to prove a weak Harnack inequality for the kinetic diffusion equation. The proof of the weak Harnack inequality for the parabolic partial differential equation can be for example found in [Mos64]. A crucial step in the proof is a uniform estimate on the measure of the logarithmic sublevel sets. To obtain these, one uses the test function $\frac{1}{u}\varphi^2$, where φ is a suitable cutoff function. This leads to an integral of the gradient of the supersolution. In the nondegenerate case this can be estimated using the following weighted Poincaré inequality.

Proposition 7.2.6. Consider the weight function $0 \neq \varphi \in C_c(\mathbb{R}^N)$, satisfying $0 \leq \varphi \leq 1$, such that the superlevel set $\{\varphi \geq a\}$ is convex for any $a \leq 1$. For $u \in W^{1,2}(\mathbb{R}^N)$ we define

$$u = \frac{1}{\|\varphi\|_{1,\mathbb{R}^N}} \int_{\mathbb{R}^N} u\varphi \mathrm{d}x,$$

then it holds

$$\int_{\mathbb{R}^N} (u(x) - u_\varphi)^2 \varphi(x) \mathrm{d}x \leq \frac{2\mathrm{diam}(\mathrm{supp}\, \varphi)^2}{\|\varphi\|_{1,\mathbb{R}^N}} \int_{\mathbb{R}^N} |\nabla u|^2 \varphi \mathrm{d}x.$$

Proof. [Mos64, Lemma 3] □

If we want to use this Poincaré inequality in our setting, we are immediately confronted with the problem that we have no control over the gradient of u in the x-direction. We want to present a Poincaré inequality which seems more suitable for the degenerate case. It is taken from [GIMV16] where it is used in the appendix to prove a gain of integrability of $\nabla_v u$.

Proposition 7.2.7. Let (t_0, v_0, x_0) and let $\varphi \in C_c^\infty(\mathbb{R})$ a cutoff function such that $\sqrt{\varphi} \in C^\infty(\mathbb{R})$, $\varphi = 1$ in $[-1, 1]$ and $\varphi = 0$ for all $|x| \geq 2$. We denote $\varphi_R(x) = \varphi(R^{-1}x)$ and define the cutoff function $\chi_{2R} \in C_c^\infty(\mathbb{R}^N)$ as

$$\chi_{2R}(t, v, x) = \prod_{i=1}^{n} \varphi_R(v_i - v_i^0)\varphi_{R^3}(x_i - x_i^0).$$

The weighted mean of a function u with respect to χ is set as

$$\tilde{u}_{2R}(t) = \frac{1}{c} \int_{\mathbb{R}^N} f(t, v, x)\chi_{2R}(v, x) \mathrm{d}v\mathrm{d}x$$

for some constant $c > 0$. Let u be a weak solution of (7.2) in some cylinder Q. If

$$Q_{3R}(t_0, v_0, x_0) = (t_0 - r^2, t_0] \times B_{3R}(v_0) \times B_{(3R)^3}(x_0) \subset Q,$$

there is a constant $C > 0$ such that it holds

$$\int_{Q_R^t(t_0,v_0,x_0)} \left| f(t, \cdot) - \tilde{f}_R \right|^2 \mathrm{d}v\mathrm{d}x \leq C \int_{Q_{3R}(t_0,v_0,x_0)} |\nabla_v f|^2 \, \mathrm{d}(t, v, x),$$

where $Q_R^t(t_0, v_0, x_0) = \{(v, x) \mid (t, v, x) \in Q_R\}$.

Proof. [GIMV16, Lemma 29] □

A problem of this Poincaré inequality is that one needs information on a cylinder of radius $3R$ to gain information on the smaller cylinder of radius R.

A Appendix

A.1 Basic notions of semigroup theory

Let X be a real Banach space and $(T(t))_{t\geq 0}$ a family of bounded and linear operators on X. We say that $T(t)$ is a strongly continuous (C_0-)semigroup if it is $T(t+s) = T(t)(s)$ for all $s,t \geq 0$ and it holds $\lim\limits_{t\to 0^+} T(t)x = x$ for all $x \in X$. We say that $(T(t))_{t\geq 0}$ is quasi-contractive if there is a constant $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq \exp(\omega t)$$

for all $t \geq 0$. The semigroup is said to be contractive if it is quasi-contractive with $\omega = 0$. The generator of a strongly continuous semigroup is the linear operator

$$Ax = \lim_{h\to 0^+} \frac{T(h)x - x}{h}$$

with domain $D(A)$, the set of all x such that this limit exists. We recall that by the Hille-Yosida theorem the generator of a strongly continuous semigroup is always closed and densely defined. A subspace $D \subset D(A)$ is said to be a core of A if the restriction $A|_D$ is closable and its closure is given by A. The following proposition gives a useful criterion to determine whether a subspace is the core of a generator.

Proposition A.1.1. Let $T(t)$ be a strongly continuous semigroup with generator A. Let $D \subset D(A)$ a linear space such that D is dense in X and $T(t)D \subset D$ for all $t > 0$, then D is a core.

Proof. [EN00, Chapter II, Proposition 1.7] □

Lemma A.1.2. Let $S(t), T(t)$ be strongly continuous semigroups with generators A and B. Suppose that $D \subset D(A)$ is a core and $A|_D \subset B$, then $S(t) = T(t)$ for all $t \geq 0$.

Proof. This follows from theorem 4.0.5, since constant sequences converge if and only if they are equal to their limit. □

Definition A.1.3. An operator $A\colon D(A) \to X$ is called accretive if

$$\|(\lambda + A)u\| \geq \lambda \|u\|$$

for all $u \in D(A)$ and every $\lambda > 0$. We say that A is m-accretive if A is accretive, densely defined and there exists $\lambda > 0$ such that $\mathcal{R}(\lambda + A) = X$. Moreover, A is quasi-(m-)accretive if there exists a constant $M > 0$ such that $A + M$ is (m-)accretive. An operator is called essentially (quasi-)(m-)accretive if it is closable and its closure is (quasi-)(m-)accretive.

Remark A.1.4. If $-A$ is accretive, one also says that A is dissipative.

From now on, we are going to consider the Banach space $X = L^p(\mathbb{R}^N)$ for $p \in (1, \infty)$. We remark that most of the definitions also make sense if X is a suitable real Banach lattice.

Definition A.1.5. A linear operator $A\colon D(A) \to L^p(\mathbb{R}^N)$ on $L^p(\mathbb{R}^N)$ is called dispersive if

$$\|(\lambda u - Au)^+\|_{p,\mathbb{R}^N} \geq \lambda \|u^+\|_{p,\mathbb{R}^N} \tag{A.1.1}$$

for all $\lambda > 0$ and all $u \in D(A)$. The operator A is called m-dispersive if A is densely defined and $\mathcal{R}(\lambda - A) = L^p(\mathbb{R}^N)$. We call A quasi (m-)dispersive if there is a positive constant $M > 0$ such that $A - M$ is (m-)dispersive. Furthermore, we say that an operator is essentially (quasi-)(m-)dispersive if it is closable and its closure is (quasi-)(m-)dispersive.

We denote by $q \in (1, \infty)$ the dual exponent to p and identify $(L^p(\mathbb{R}^N))' = L^q(\mathbb{R}^N)$. We define the set $J(u) = \{f \in L^q(\mathbb{R}^N) \mid \|f\|_{q,R^N} \leq 1, f \geq 0$ and $\langle u, f\rangle = \|u^+\|_{p,\mathbb{R}^N}\}$. We note that if it is $u \in L^p(\mathbb{R}^N)$, then it holds $\|u^+\|_{p,\mathbb{R}^N}^{-\frac{p}{q}} (u^+)^{p-1} \in J(u)$.

Proposition A.1.6. If for all $u \in D(A)$ there is some $f \in J(u)$ such that

$$\langle Au, f\rangle \leq 0,$$

then A is dispersive. Consequently, if it holds $\langle Au, f\rangle \leq M\langle u, f\rangle$ for some $M > 0$, the operator A is quasi-dispersive.

Proof. [BFR17, Proposition 11.12] □

Definition A.1.7. A semigroup $(T(t))_{t\geq 0}$ on $L^p(\mathbb{R}^N)$ is called positive if $T(t)f \geq 0$ for all nonnegative $f \in L^p(\mathbb{R}^N)$ and all $t \geq 0$. The resolvent $R(\lambda, A)$ is called positivity preserving if $R(\lambda, A)f \geq 0$ for all $0 \leq f \in L^p(\mathbb{R}^N)$ and any $\lambda \in \rho(A)$.

Proposition A.1.8. Let $T(t)$ be a strongly continuous semigroup on $L^p(\mathbb{R}^N)$. It is positive if and only if the resolvent is positivity preserving.

Proof. [Jac01, Lemma 4.6.5] □

Theorem A.1.9. *Let $A\colon D(A) \to L^p(\mathbb{R}^N)$ be a linear operator and $\omega \in \mathbb{R}$. The following two statements are equivalent:*

(i) A is the generator of quasi-contractive positive C_0-semigroup.

(ii) A is quasi-dispersive and $\mathcal{R}(\lambda - A) = L^p(\mathbb{R}^N)$ for some $\lambda > 0$.

Proof. Using the scaling argument $S(t) = \exp(-\omega t)T(t)$, this theorem is a consequence of [Phi62, Theorem 2.1]. □

Proposition A.1.10. If $A\colon D(A) \to L^p(\mathbb{R}^N)$ is a dispersive semigroup operator, then $-A$ is an accretive operator. Conversely, if A is dispersive, then $-A$ is accretive.

Proof. This is a consequence of [Phi62, Corollay] and [Phi62, Lemma 2.3]. We note that on Banach lattice $L^p(\mathbb{R}^N)$ the map

$$[f,g] := \frac{1}{\|f\|_{p,\mathbb{R}^N}^{p-2}} \int_{\mathbb{R}^N} f\,|g|^{p-2}\,g \mathrm{d}x$$

defines a semi inner product. Moreover, one readily verifies that $[f,g] = [f,g^+] - [f,(-g)^+]$ so that the assumption of [Phi62, Lemma 2.3] is fulfilled. Finally, the claim follows by [Phi62, Lemma 2.3] noting that an operator A is dissipative if and only if $-A$ accretive. □

Remark A.1.11. The latter proposition shows that in the Banach lattice $L^p(\mathbb{R}^N)$ the concepts of dispersiveness and dissipativeness coincide. An operator A is called dissipative if and only if $-A$ is accretive. So that A is dissipative if and only if A is dispersive. We note that sometimes dispersiveness may be easier to verify, since one only has to deal with nonnegative functions.

Lemma A.1.12. Let A be closable and (quasi-)dispersive operator then the closure is (quasi-)dispersive as well. The same result holds for accretiveness of operators.

Proof. The first claim is a consequence of [ACK82, Theorem 2.4] and the second one is a part of the proof of [EN00, Proposition 3.14]. □

Let us now collect some results regarding solution of the Cauchy problem

$$\begin{cases} \partial_t u = Au, \quad t > 0 \\ u(0) = u_0 \end{cases} \tag{A.1.2}$$

for $u_0 \in L^p(\mathbb{R}^N)$.

Definition A.1.13. We call a function $u \in C^1([0, \infty); L^p(\mathbb{R}^N))$ such that it holds $u(t) \in D(A)$ and $\partial_t u \in L^p(\mathbb{R}^N)$ for all $t > 0$ a strong solution of the Cauchy problem (A.1.2) if for all $t > 0$ it holds $\partial_t u = Au$ and $u(0) = u_0$.

Definition A.1.14. A function $u \in C([0, \infty); L^p(\mathbb{R}^N))$ is called mild solution if it holds

$$\int_s^t u(\tau) \mathrm{d}\tau \in D(A)$$

and

$$u(t) - u(s) = A \int_s^t u(\tau) \mathrm{d}\tau$$

for all $0 \leq s \leq t$.

Definition A.1.15. A function $u \in C([0, \infty); L^p(\mathbb{R}^N))$ is called weak solution if for all $\varphi \in D(A')$ the function $t \mapsto \langle u(s), \varphi \rangle$ is absolutely continuous on every interval $[0, T]$ and it holds

$$\int_0^T \langle u(s), \varphi \rangle \partial_t \psi(s) \mathrm{d}s = \langle u_0, \varphi \rangle \psi(0) - \int_0^T \langle u, A' \varphi \rangle \psi(s) \mathrm{d}s$$

for all $\varphi \in C_c^\infty(\mathbb{R}^N)$ and all $\psi \in C_c^\infty([0, T))$.

Proposition A.1.16. Let A be the generator of a C_0-semigroup, then there is a unique mild solution for all $u_0 \in L^p(\mathbb{R}^N)$. It is a strong solution if and only if $u_0 \in D(A)$. In this case the solution is given by $t \to T(t)u_0$.

Proof. [EN00, Chapter II, Section 6, Proposition 6.2 and 6.4] □

Proposition A.1.17. If A is the generator of a C_0-semigroup, then a function $u \in C([0, \infty); L^p(\mathbb{R}^N))$ is a mild solution to the Cauchy problem (A.1.2) if and only if it is also a weak solution.

Proof. [Bal77, Theorem] □

A.2 Fractional Sobolev spaces

In this section we collect some properties of fractional Sobolev spaces.

Definition A.2.1. Let $s > 0$. We define the fractional Sobolev space of order s as

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} \left(1 + |\xi|^{2s}\right) |[\mathcal{F}u](\xi)|^2 \,\mathrm{d}\xi < \infty \right\}$$

For every $u \in H^s(\mathbb{R}^N)$ we define

$$\|u\|_{s,2,\mathbb{R}^N}^2 = \|u\|_{2,\mathbb{R}^N}^2 + 2C(N,s)^{-1} \int_{\mathbb{R}^N} |\xi|^{2s} |[\mathcal{F}u](\xi)|^2 \,\mathrm{d}\xi,$$

where

$$C(N,s)^{-1} = \int_{\mathbb{R}^N} \frac{1 - \cos(\xi_1)}{|\xi|^{n+2s}} \mathrm{d}\xi.$$

Definition A.2.2. Given $s \in [0,1]$, we define the fractional Laplacian $(-\Delta)^s : \mathcal{S} \to L^2(\mathbb{R}^N)$ as

$$(-\Delta)^s u(x) = \mathcal{F}^{-1}(\xi \mapsto |\xi|^{2s} [\mathcal{F}u](\xi))(x)$$

for all $x \in \mathbb{R}^N$. In particular, it is $(-\Delta)^0 = \mathrm{Id}$ and $(-\Delta)^1 = -\Delta$. We are going to denote by $D^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ the fractional derivative of order $\alpha \in [0,2]$. Clearly, $D^\alpha u$ is well-defined for any $u \in H^\alpha(\mathbb{R}^N)$. If $u = u(t,v,x)$, the fractional derivative in the position variable is given by

$$D_x^\alpha u = (-\Delta_x)^{\frac{\alpha}{2}} u = \mathcal{F}_x^{-1}(k \mapsto |k|^\alpha [\mathcal{F}_x u](k)),$$

where $\mathcal{F}_x$ denotes Fourier transform only in the variable x. The fractional derivative with respect to v is given similarly.

Lemma A.2.3. Let $u \in H^{2s}(\mathbb{R}^N)$, then

$$\langle D^s u, D^s u \rangle = \langle D^{2s} u, u \rangle.$$

Proof. Using the theorem of Plancherel twice, we obtain

$$\langle D^s u, D^s u \rangle = \langle |\xi|^s \mathcal{F}(u), |\xi|^s \mathcal{F}(u) \rangle = \langle |\xi|^{2s} \mathcal{F}(u), \mathcal{F}(u) \rangle = \langle D^{2s} u, u \rangle.$$

□

Lemma A.2.4. If $u \in \mathcal{S}(\mathbb{R}^N)$, it holds

$$\langle \partial_{x_j} u, u \rangle \leq \left\| D^{1-\gamma} u \right\|_{2,\mathbb{R}^N} \|D^\gamma u\|_{2,\mathbb{R}^N}$$

for all $0 \leq \gamma \leq 1$ and any $j = 1, \ldots, N$.

Proof. Let $u \in \mathcal{S}(\mathbb{R}^N)$, $\gamma \in [0,1]$ and $i \in \{1, \ldots, N\}$, then by the theorem of Plancherel it holds

$$\langle \partial_{x_j} u, u \rangle = \langle i\xi_j \mathcal{F}(u), \mathcal{F}(u) \rangle \leq \langle |\xi| \, \mathcal{F}(u), \mathcal{F}(u) \rangle = \langle |\xi|^{1-\gamma} \, \mathcal{F}(u), |\xi|^{\gamma} \, \mathcal{F}(u) \rangle = \langle D^{1-\gamma} u, D^{\gamma} u \rangle.$$

The Cauchy-Schwarz inequality shows the claim. $\square$

A.3 Approximation and smoothing of functions

Definition A.3.1. We say that a sequence of integrable functions $(\varphi_k)_{k\in\mathbb{N}} \subset L^1(\mathbb{R}^N)$ is a Dirac sequence if

(i) for all $k \in \mathbb{N}$ it is $\varphi_k \geq 0$,

(ii) for all $k \in \mathbb{N}$ it holds $\int_{\mathbb{R}^N} \varphi_k \mathrm{d}x = 1$,

(iii) for all $\delta > 0$ it holds

$$\lim_{k\to\infty} \int_{\mathbb{R}^N \setminus B_\delta(0)} \varphi_k \mathrm{d}x = 0.$$

A sequence of measurable functions $(\varphi_k)_{k\in\mathbb{N}}$, $\varphi_k \colon \mathbb{R}^{2N} \to \mathbb{R}$ for all $k \in \mathbb{N}$ is called generalized Dirac sequence if (i) holds and

(ii)' for all $k \in \mathbb{N}, x \in \mathbb{R}^N$ it holds $\int_{\mathbb{R}^N} \varphi_k(x, y)\mathrm{d}y = 1$,

(iii)' for all $\delta > 0, x \in \mathbb{R}^N$ and every sequence $x_k \to x$ it holds

$$\lim_{k\to\infty} \int_{\mathbb{R}^N \setminus B_\delta(x)} \varphi_k(x_k, y)\mathrm{d}y = 0.$$

Remark A.3.2. If $\varphi_k(\cdot,\cdot)$ is a generalized Dirac sequence, $\varphi_k(x,\cdot)$ defines a Dirac sequence for all $x \in \mathbb{R}^N$.

Proposition A.3.3. Let φ_k be a generalized Dirac sequence, $(f_k)_{k\in\mathbb{N}} \subset C_b(\mathbb{R}^N)$ a bounded sequence, $x_0 \in \mathbb{R}^N$ and a sequence $x_k \to x_0$. We suppose that for all $\varepsilon > 0$ there is $K \in \mathbb{N}$ and $\delta > 0$ so that $|f_k(y) - f(x_0)| < \varepsilon$ for all $k \geq K$ and any $y \in \mathbb{R}^N$ with $|y - x_0| < \delta$. Under this assumption it holds

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \varphi_k(x_k, y) f_k(y)\mathrm{d}y = f(x_0).$$

In particular, choosing $f_k = f$ and any $x \in \mathbb{R}^N$, it holds

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \varphi_k(x_k, y) f(y)\mathrm{d}y = f(x).$$

for every sequence $x_k \to x \in \mathbb{R}^N$.

Proof. Let $\varepsilon > 0$, then exists $\delta > 0$ and $K \in \mathbb{N}$ such that $|y - x_0| < \delta$ implies $|f_k(y) - f(x_0)| < \varepsilon$ for all $k \geq K$. Furthermore, we can choose K so large that for all $k \geq K$ it is

$$\int_{\mathbb{R}^N \setminus B_\delta(x_0)} \varphi_k(x_k, y)\mathrm{d}y \leq \varepsilon.$$

We conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} \varphi_k(x_k,y) f_k(y)\mathrm{d}y - f(x_0)\right| &\le \int_{\mathbb{R}^N} \varphi_k(x_k,y)\left|f_k(y)-f(x_0)\right|\mathrm{d}y \\ &= \int_{B_\delta(x_0)} \varphi_k(x_k,y)\left|f_k(y)-f(x_0)\right|\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^N\setminus B_\delta(x_0)} \varphi_k(x_k,y)\left|f_k(y)-f(x_0)\right|\mathrm{d}y \\ &\le \varepsilon + 2\left\|f_k\right\|_{\infty,\mathbb{R}^N}\varepsilon,\end{aligned}$$

which shows the first claim. The second claim follows from the first one noting that if $f_k = f$, the additional assumption is just the continuity of f in any point $x \in \mathbb{R}^N$. □

Proposition A.3.4. Let $(\varphi_k)_{k\in\mathbb{N}}$ be a Dirac sequence. Let $p \in [1,\infty)$ and $f \in L^p(\mathbb{R}^N)$, then $\varphi_k * f \to f$ in $L^p(\mathbb{R}^N)$ as $k \to \infty$.

Proof. [Alt16, Theorem 4.15] □

Let us introduce the standard mollifier $\omega \in C_c^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} \omega(x)\mathrm{d}x = 1$ and $\operatorname{supp}\omega \subset \overline{B}_1(0)$. Given $\varepsilon > 0$, we define $\omega_\varepsilon(x) = \varepsilon^{-n}\omega(\varepsilon^{-1}x)$. If $\varepsilon = \frac{1}{k}$ for some $k \in \mathbb{N}$, we write ω_k instead. In particular, if $\omega \ge 0$, then the sequence $(\omega_\varepsilon)_{\varepsilon>0}$ is a Dirac sequence.

Theorem A.3.5 (Friedrich's approximation theorem)**.** *Let A be a second order differential operator as in equation* (A.1.1) *satisfying the assumptions (A1) and (A2). Let $u \in D(\mathcal{A}_p)$, then*

$$Au * \omega_\varepsilon \to Au$$

in $L^p(\mathbb{R}^N)$ as $\varepsilon \to 0$.

Proof. This is a consequence of [Kat72, Lemma 2] by using the partition $\mathbb{R}^N = \bigcup_{i=1}^\infty B_i(0) \setminus \overline{B_i(0)}$. □

We further introduce the cutoff functions $\eta_\varepsilon(x) = \eta(\varepsilon x)$ where $\eta \in C_c^\infty(\mathbb{R}^N)$, $0 \le \eta \le 1$ such that $\eta = 1$ on $\bar{B}_1(0)$ and $\eta(x) = 0$ for all $|x| \ge 2$. Furthermore, $|\nabla\eta|$ is bounded by some constant. The following lemma collects some useful properties.

Lemma A.3.6. Given $k \in \mathbb{N}$, we set $\eta_k(x) = \eta(\frac{x}{k})$ for all $x \in \mathbb{R}^N$. Let $p \in [1, \infty)$. It holds

(i) $\eta_k \to 1$ pointwise on $\mathbb{R}^N$,

(ii) $|\nabla \eta_k| \leq \frac{c_1}{k}$ for some constant $c_1 > 0$,

(iii) $\nabla \eta_k \to 0$ uniformly on $\mathbb{R}^N$,

(iv) $\eta_k f \to f$ in $L^p(\mathbb{R}^N)$ for all $f \in L^p(\mathbb{R}^N)$.

Proof. The first property follows by definition of η_k, the second by differentiation, the third is an immediate consequence of the second property and the last property follows from the theorem of dominated convergence. $\square$

B Technical results

Lemma B.0.1. Let $A \in \mathbb{R}^{N\times N}$ be symmetric and positive semidefinite, then for all $x, y \in \mathbb{R}^N$ it holds

$$|\langle Ax, y\rangle| \leq \sqrt{\langle Ax, x\rangle}\sqrt{\langle Ay, y\rangle}.$$

Proof. [Wal16, Proposition 7.10] $\square$

Lemma B.0.2. Let $A, B \in \mathbb{R}^{N\times N}$ be symmetric matrices such that A is positive semidefinite. Then for all $x, y \in \mathbb{R}^N$ it holds

$$|\langle ABx, y\rangle| \leq \sqrt{\operatorname{tr} BAB}\,|x|\,\sqrt{\langle Ay, y\rangle}.$$

Proof. Let $x, y \in \mathbb{R}^N$, then by the Cauchy-Schwarz inequality B.0.1 and the compatibility of the Frobenius matrix norm and Euclidean norm we deduce

$$|\langle ABx, y\rangle| = \left|\langle A^{\frac{1}{2}}Bx, A^{\frac{1}{2}}y\rangle\right| \leq \left|A^{\frac{1}{2}}y\right|\left|A^{\frac{1}{2}}Bx\right| \leq \sqrt{\langle Ay, y\rangle}\sqrt{\operatorname{tr}(BA^{\frac{1}{2}}A^{\frac{1}{2}}B)}\,|x|\,.$$

$\square$

Lemma B.0.3. Let $A \in \mathbb{R}^{N\times N}$ be symmetric positive semidefinite and let $B \in \mathbb{R}^{N\times N}$ be symmetric negative semidefinite. Then $\operatorname{tr}(AB) \leq 0$.

Proof. Define $A_\varepsilon = A + \varepsilon\,\mathrm{Id}_N$. Note that from

$$\langle A_\varepsilon^{\frac{1}{2}}BA_\varepsilon^{\frac{1}{2}}x, x\rangle = \langle BA_\varepsilon^{\frac{1}{2}}x, A_\varepsilon^{\frac{1}{2}}x\rangle \leq 0$$

for all $x \in \mathbb{R}^N$ it follows that $A_\varepsilon^{\frac{1}{2}}BA_\varepsilon^{\frac{1}{2}}$ is symmetric negative semidefinite. Consequently,

$$\operatorname{tr}(A_\varepsilon B) = \operatorname{tr}\left(A_\varepsilon^{\frac{1}{2}}A_\varepsilon^{\frac{1}{2}}B\right) = \operatorname{tr}\left(A_\varepsilon^{\frac{1}{2}}BA_\varepsilon^{\frac{1}{2}}\right) \leq 0.$$

The lemma follows by recalling the continuity of the trace and taking the limit $\varepsilon \to 0$. $\square$

Lemma B.0.4. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $g\colon \mathbb{R}^n \to [0,\infty)$ be measurable functions and $A \in \mathbb{R}^{N\times N}$ be a symmetric positive semidefinite matrix such that $\int_{\mathbb{R}^N} \langle Af, f\rangle g \mathrm{d}x, \int_{\mathbb{R}^n} g \mathrm{d}x < \infty$. Then it holds that

$$\left\langle A \int_{R^n} fg\mathrm{d}x, \int_{\mathbb{R}^n} fg\mathrm{d}x \right\rangle \leq \int_{\mathbb{R}^n} \langle Af, f\rangle g\mathrm{d}x \int_{\mathbb{R}^n} g\mathrm{d}x.$$

Proof. We consider the case that $A = \mathrm{Id}_n$ first. We estimate

$$\left\| \int_{\mathbb{R}^N} fg\mathrm{d}x \right\|^2 = \sum_{i=1}^n \left(\int_{\mathbb{R}^N} f_i g\mathrm{d}x \right)^2 \leq \sum_{i=1}^n \int_{\mathbb{R}^N} f_i^2 g\mathrm{d}x \int g\mathrm{d}x = \int_{\mathbb{R}^n} \|f\|^2 g\mathrm{d}x \int_{\mathbb{R}^n} g\mathrm{d}x$$

by applying the Cauchy-Schwarz inequality to the functions $f_i\sqrt{g}$ and $\sqrt{g}$. For arbitrary symmetric positive semidefinite A we proceed as follows. Since A is symmetric and positive semidefinite, there exists the matrix root $A^{\frac{1}{2}}$ of A. Further, it holds

$$\begin{aligned}
\left\langle A^{\frac{1}{2}} \int_{R^n} fg\mathrm{d}x, A^{\frac{1}{2}} \int_{\mathbb{R}^n} fg\mathrm{d}x \right\rangle &= \left\langle A^{\frac{1}{2}} \int_{R^n} fg\mathrm{d}x, A^{\frac{1}{2}} \int_{\mathbb{R}^n} fg\mathrm{d}x \right\rangle \\
&= \left\langle \int_{R^n} A^{\frac{1}{2}} fg\mathrm{d}x, \int_{\mathbb{R}^n} A^{\frac{1}{2}} fg\mathrm{d}x \right\rangle \\
&\leq \int_{\mathbb{R}^n} \|A^{\frac{1}{2}} f\|^2 g\mathrm{d}x \int_{\mathbb{R}^n} g\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \langle A^{\frac{1}{2}} f, A^{\frac{1}{2}} f\rangle g\mathrm{d}x \int_{\mathbb{R}^n} g\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \langle Af, f\rangle g\mathrm{d}x \int_{\mathbb{R}^n} g\mathrm{d}x,
\end{aligned}$$

since $A^{\frac{1}{2}} A^{\frac{1}{2}} = A$ and due to the symmetry of A, it holds $\langle A^{\frac{1}{2}} x, A^{\frac{1}{2}} x\rangle = \langle Ax, x\rangle$ for all $x \in \mathbb{R}^n$. This shows the claim. $\square$

The following lemma is a classical statement from probability theory. Namely it deals with the multivariate normal distribution.

Lemma B.0.5. Let $A \in \mathbb{R}^{N\times N}$ be symmetric positive definite matrix. Then

$$\sqrt{\frac{\det(A)}{(2\pi)^N}} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2} \langle Ax, x\rangle \right) \mathrm{d}x = 1$$

and

$$\int_{\mathbb{R}^N} \exp\left(-\frac{1}{2} \langle Ax, x\rangle \right) x\mathrm{d}x = 0.$$

Lastly, it holds

$$\sqrt{\frac{\det(A)}{(2\pi)^N}}\int_{\mathbb{R}^N}\exp\left(-\frac{1}{2}\langle Ax,x\rangle\right)x_ix_j\mathrm{d}x=(A^{-1})_{ij}$$

for all $i,j\in\{1,\dots,N\}$.

Proof. [Gut09, Chapter 5] □

Lemma B.0.6. Let $\mathcal{C}\in\mathbb{R}^{N\times N}$ be a symmetric positive definite and $(t,x)\in\mathbb{R}^N$. Let $B\in\mathbb{R}^{N\times N}$ be as in section 3.1 and denote $E(t)=\exp(-tB^T)$. There exists a constant $c_1=c_1(B,\mathcal{C},R)>0$ and constant $R=R(x,B)>0$ such that

$$\langle\mathcal{C}\delta_{\frac{1}{\sqrt{s}}}(x-E(s)\xi),\delta_{\frac{1}{\sqrt{s}}}(x-E(s)\xi)\rangle\geq\frac{c_1}{s}|\xi|^2$$

for every $(s,\xi)\in(0,1)\times B_R^c(x)$.

Proof. Let $R>0$ and $(s,\xi)\in(0,1)\times B_R^c(x)$. We write

$$\begin{aligned}\langle\mathcal{C}\delta_{\frac{1}{\sqrt{s}}}(x-E(s)\xi),\delta_{\frac{1}{\sqrt{s}}}(x-E(s)\xi)\rangle&=\langle\mathcal{C}\delta_{\frac{1}{\sqrt{s}}}E(s)(\xi-E(-s)x),\delta_{\frac{1}{\sqrt{s}}}E(s)(\xi-E(-s)x)\rangle\\&=\langle\mathcal{C}E(1)\delta_{\frac{1}{\sqrt{s}}}(\xi-E(-s)x),E(1)\delta_{\frac{1}{\sqrt{s}}}(\xi-E(-s)x)\rangle\\&=\langle E(1)^T\mathcal{C}E(1)\delta_{\frac{1}{\sqrt{s}}}(\xi-E(-s)x),\delta_{\frac{1}{\sqrt{s}}}(\xi-E(-s)x)\rangle\end{aligned}$$

by corollary 3.1.10. Since $E(1)^T\mathcal{C}E(1)$ is positive definite, there exists a constant $\tilde{c}_1$ such that

$$\begin{aligned}\langle E(1)^T\mathcal{C}E(1)\delta_{\frac{1}{\sqrt{s}}}(\xi-E(-s)x),\delta_{\frac{1}{\sqrt{s}}}(\xi-E(-s)x)\rangle&\geq\tilde{c}_1\left|\delta_{\frac{1}{\sqrt{s}}}(\xi-E(-s)x)\right|^2\\&\geq\frac{\tilde{c}_1}{s}|\xi-E(-s)x|^2\\&\geq\frac{\tilde{c}_1}{s}\left(|\xi|^2-|E(-s)x|^2\right)\end{aligned}$$

for all $s\in(0,1)$. If we choose $R\geq2\sup_{s\in(0,1)}|E(-s)x|$ and $c_1=\frac{3\tilde{c}_1}{4}$, this shows

$$\langle\mathcal{C}\delta_{\frac{1}{\sqrt{s}}}(x-E(s)\xi),\delta_{\frac{1}{\sqrt{s}}}(x-E(s)\xi)\rangle\geq\frac{\tilde{c}_1}{s}\left(|\xi|^2-|E(-s)x|^2\right)\geq\frac{\tilde{c}_1}{s}\left(|\xi|^2-\frac{R^2}{4}\right)\geq\frac{3\tilde{c}_1}{4s}|\xi|^2$$

and hence the claim. □

Lemma B.0.7. Let X be a Banach space and $(T_n)_{n\in\mathbb{N}}$ a sequence of linear and bounded operators such that $\sup_{n\in\mathbb{N}}\|T_n\|<\infty$. Let $D\subset X$ be dense in X. If $T_nf\to f$ as $n\to\infty$ for all $f\in D$, then $T_nf\to f$ for all $f\in X$.

Proof. Let $\varepsilon > 0$ and $f \in X$. We choose any $g \in D$ such that $\|f - g\| < \varepsilon$ and $N \in \mathbb{N}$ such that $\|T_n g - g\| < \varepsilon$. It follows that

$$\|T_n f - f\| \leq \|T_n f - T_n g\| + \|T_n g - g\| + \|g - f\| < (\sup_{n\in\mathbb{N}} \|T_n\| + 2)\varepsilon$$

for all $n \geq N$. This shows the claimed convergence. □

Proposition B.0.8. There exists a cylinder $Q = (0,1) \times B_R(0) \times B_R(0)$ such that the truncated fundamental solution Φ_k from the proof of proposition 7.2.5 is a weak supersolution of

$$\partial_t u + v \cdot \nabla_x u = \Delta_v u.$$

Proof. Let $k \in \mathbb{N}$. We recall the definition of the truncated fundamental solution. It is

$$\Phi_k(t,v,x) = \begin{cases} \Gamma(t,v,x) & t \geq \frac{1}{k} \\ \Gamma\left(\frac{1}{k}, v, x\right) & t \leq \frac{1}{k}, \end{cases}$$

where Γ denotes the fundamental solution of the Kolmogorov equation $\partial_t u + v \cdot \nabla_x u = \Delta_v u$. We immediately see that Φ_k is a solution and in particular a supersolution of the Kolmogorov equation for $t \geq \frac{1}{k}$. For $t \leq \frac{1}{k}$ we calculate

$$\begin{aligned}
\nabla_x \Phi_k\left(\frac{1}{k}, v, x\right) &= \Phi_k\left(\frac{1}{k}, v, x\right)\left[-6k^3 x + 3k^2 v\right], \\
\nabla_v \Phi_k\left(\frac{1}{k}, v, x\right) &= \Phi_k\left(\frac{1}{k}, v, x\right)\left[-2kv + 3k^2 x\right], \\
v \cdot \nabla_x \Phi_k\left(\frac{1}{k}, v, x\right) &= \Phi_k\left(\frac{1}{k}, v, x\right)\left[-6k^3 \langle v, x\rangle + 3k^2 |v|^2\right], \\
\Delta_v \Phi_k\left(\frac{1}{k}, v, x\right) &= \Phi_k\left(\frac{1}{k}, v, x\right)\left[-2kn + 4k^2 |v|^2 - 12k^3 \langle x, v\rangle + 9k^4 |x|^2\right].
\end{aligned}$$

Combining these equations shows that

$$\begin{aligned}
v \cdot \nabla_x \Phi_k\left(\frac{1}{k}, v, x\right) - \Delta_v \Phi_k\left(\frac{1}{k}, v, x\right) &= \Phi_k\left(\frac{1}{k}, v, x\right)\left[2kn - k^2 |v|^2 + 6k^3 \langle x, v\rangle - 9k^4 |x|^2\right] \\
&\geq \Phi_k\left(\frac{1}{k}, v, x\right)\left[kn - 4k^2 |v|^2 + kn - 12k^4 |x|^2\right] \geq 0
\end{aligned}$$

for all $(v,x) \in B_{\sqrt{\frac{n}{4}}}(0) \times B_{\sqrt{\frac{n}{12}}}(0)$, since $k \geq 1$. We choose $R = \sqrt{\frac{n}{12}}$. Finally, since $\Phi_k \in C((0,1) \times B_R(0) \times B_R(0)) \cap C^\infty(\left((0,\frac{1}{k}) \cup (\frac{1}{k}, 1)\right) \times B_R(0) \times B_R(0))$ is a supersolution on the set $\left((0,\frac{1}{k}) \cup (\frac{1}{k}, 1)\right) \times B_R(0) \times B_R(0)$, we conclude that Φ_k is a weak supersolution of the Kolmogorov equation in $Q = (0,1) \times B_R(0) \times B_R(0)$, too. □

C Notation

Given a function $u\colon \mathbb{R}^{N+1} \to \mathbb{R}$, the first argument is considered as time, while the following arguments represent the space coordinates. ∇ denotes only the space gradient and ∇_v the gradient with respect to the variables v. Depending on the context, differential operators are either viewed as maps on suitable function spaces or as vector fields in $\mathbb{R}^{N+1}$ with the same convention on the order of space and time as in the argument of a function. Exploiting the notation of a scalar product, we define $\langle v, \nabla \rangle = \sum_{i=1}^{N} v_i \partial_{x_i}$ for all $v \in \mathbb{R}^N$.

We are going to consider also linear operators, i.e. linear maps $A\colon D(A) \to X$ in some Banach space X. If such an operator is closable, we denote by $\overline{A}$ its closure. If $X = L^p(\mathbb{R}^N)$ for some $p \in (1, \infty)$, we write $\overline{A}^p$ to emphasize the p-dependence of the closure. If A has nonempty resolvent, we denote by $R(\lambda, A) = (\lambda - A)^{-1}$ the resolvent map.

Nomenclature

$(C_b(\Omega), \|\cdot\|_{\infty,\mathbb{R}^N})$	Space of continuous and bounded functions
$(L^p(\Omega), \|\cdot\|_{p,\Omega})$	Lebesgue spaces of p-integrable functions
$(W^{k,p}(\Omega), \|\cdot\|_{k,p,\Omega})$	Sobolev space of order $k \in \mathbb{R}$ and integrability p
$\lvert\cdot\rvert$	Euclidean norm in $\mathbb{R}^N$
$\mathcal{D}'(\Omega)$	The set of all distributions in Ω
$\mathrm{d}\mathcal{S}(x)$	The surface measure on the boundary of a set with smooth boundary
Id_N	Identity matrix in $\mathbb{R}^{N\times N}$
$\langle \cdot, \cdot \rangle$	Depending on the input, this denotes either the Euclidean scalar product, the dual pairing of a vector space and its dual or the canonical dual pairing of $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$
$\mathcal{N}$	The kernel of an linear operator
$\mathbb{1}$	Indicator function or a vector in $\mathbb{R}^N$ of ones
$\|\cdot\|_p$	Operator norm for an arbitrary matrix induced by the p-norm on $\mathbb{R}^N$
∂_{x_i}	Partial differentiation with respect to the coordinate x_i
$\mathcal{R}$	The range of an linear operator
$\mathcal{S}$	Space of Schwartz functions
$C^{1,2}((0,T) \times \Omega)$	Vector space of functions that are one time continuously differentiable in time and twice in space
$C_c^\infty(\Omega)$	Vector space of all smooth functions with compact support
Df	Jacobi matrix of a suitable function $f\colon U \subset \mathbb{R}^N \to \mathbb{R}^M$

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Erklärung

Ich erkläre, dass ich die Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Ulm, den . . . .

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