Critical trajectories in kinetic geometry

Prototypical PDE – Kolmogorov (1934) [$\exists$ fund. solution]

$$(\partial_t + v \cdot \nabla_x) f - \Delta_v f = S$$

$f = f(t, x, v)$ particle distribution

$f : \mathbb{R}^{1+2d} \to \mathbb{R}$

Scaling invariance

$$\lambda \mapsto (\lambda^2 t, \lambda^3 x, \lambda v)$$

Translation invariance

$$(t_0, x_0, v_0) \circ (t, x, v) = (t + t_0, x + x_0 + t v_0, v + v_0)$$

Goal:

Understand the regularity of weak solutions.

Energy estimate

$\varphi \in C_c^\infty((-2,1]\times B_2\times B_2)$ with $\varphi = 1$ in $Q_1 = (-1,0]\times B_1\times B_1$

$Q_2 = (-2,0]\times B_2\times B_2$.

Fix $t\in(-1,0]$. Multiply by $f\varphi^2$ & integrate on $(-2,t]\times\mathbb{R}^{2d}$

$\int \frac{1}{2}(\partial_t + v\cdot\nabla_x)(f\varphi)^2 - \frac{1}{2}\int f^2(\partial_t+v\cdot\nabla_x)\varphi^2 + \int|\nabla_v(\varphi f)|^2 - \int f^2|\nabla_v\varphi|^2 = 0$

$$\int |f\varphi(t,\cdot)|^2\,d(x,v) + \int_{-2}^{t}\int |\nabla_v(\varphi f)|^2\,d(t,x,v) \le C\int_{Q_2} f^2\,d(t,x,v)$$

$$\Rightarrow \sup_{t\in(-1,0]}\int_{B_1\times B_1}|f(t,\cdot)|^2\,d(x,v) + \int_{Q_1}|\nabla_v f|^2\,d(t,x,v) \le C\int_{Q_2}|f|^2\,d(t,x,v)$$

Natural solution space: $L^\infty_t L^2_{x,v}\cap L^2_{t,x}H^1_v$.

Questions: Regularity properties, especially in x?

Have control via $f\in L^2_{t,x,v}$, $\nabla_v f\in L^2_{t,x,v}$, $(\partial_t + v\cdot\nabla_x)f = \nabla_v\cdot S$

$S\in L^2_{t,x,v}$

i.e. $\in L^2_{t,x}H^{-1}_v$

x - Regularity? Hörmander / Kolmogorov ...

Kinetic trajectories

We know regularity of f along $\underbrace{\partial_t + v\cdot\nabla_x}_{\text{order } -1}$ & $\underbrace{\nabla_v}_{\text{order } 1}$.

Thinking about the proof of the Poincaré inequality we need to control differences of the function at two different points while using only control of f via $\partial_t + v\cdot\nabla_x$ & ∇_v

$$f(t_1,x_1,v_1) - f(t_0,x_0,v_0) = \int_0^1 \frac{d}{dr} f(\gamma(r))\, dr$$

$\gamma : [0,1] \to \mathbb{R}^{1+2d}$ with $\gamma(0) = (t_0,x_0,v_0)$, $\gamma(1) = (t_1,x_1,v_1)$

$$= \int_0^1 \dot\gamma_t [\partial_t f](\gamma(r)) + \dot\gamma_x \cdot [\nabla_x f](\gamma(r)) + \dot\gamma_v \cdot [\nabla_v f](\gamma(r))\, dr$$

$$= \int_0^1 \dot\gamma_t [(\partial_t + v\cdot\nabla_x) f](\gamma(r)) + \dot\gamma_v \cdot [\nabla_v f](\gamma(r))\, dr$$

if $\boxed{\dot\gamma_x = \dot\gamma_t \gamma_v}$ (equivalently γ is integral curve to vector fields $\partial_t + v\cdot\nabla_x$ & $\nabla_v f$)

Desired properties: good scaling & works with $(\partial_t + v\cdot\nabla_x) f = \nabla_v \cdot S$.

Construction of $\gamma = (\gamma_t, \gamma_x, \gamma_v)$

Ansatz: • $\gamma_t(v) = t_0 + m_0 r$ with $m_0 = t_1 - t_0 \neq 0$

• $\dot{\gamma}_v = \ddot{g}_1(v)\, m_1 + \ddot{g}_2(v)\, m_2$ for $g_1, g_2 \in C^1([0,1]) \cap C^2((0,1))$

Integrate once

$$\gamma_v(v) = \dot{g}_1(v)\, m_1 + \dot{g}_2(v)\, m_2 + v$$

use $\dot{\gamma}_x = \dot{\gamma}_t \gamma_v = (t_1 - t_0)\left[\dot{g}_1(v)\, m_1 + \dot{g}_2(v)\, m_2 + v_0\right]$

(kinetic trajectory)

and integrate

$$\gamma_x(v) = (t_1 - t_0)\, g_1(v)\, m_1 + (t_1 - t_0)\, g_2(v)\, m_2 + (t_1 - t_0) r v_0 + x_0.$$

$$\begin{pmatrix} \gamma_x(v) \\ \gamma_v(v) \end{pmatrix} = D_{t_1 - t_0} \underbrace{N(v)} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & (t_1 - t_0) r \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$

$$N(v) = \begin{pmatrix} g_1 & g_2 \\ \dot{g}_1 & \dot{g}_2 \end{pmatrix} \qquad = \mathcal{E}_{t_1 - t_0}(v)$$

Wronskian

$$D_\delta = \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}$$

Endpoint cond.

$$\mathcal{D}_{t_1-t_0} W(1) \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \mathcal{E}_{t_1-t_0}(1) \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} x_1 \\ v_1 \end{pmatrix}.$$

If $W(1)$ invertible, then

$$\gamma(r) = \begin{pmatrix} \gamma_t(r) \\ \gamma_x(r) \\ \gamma_v(r) \end{pmatrix} = \begin{pmatrix} t_0 + (t_1-t_0) r \\ \\ A_{t_1-t_0}(r) \begin{pmatrix} x_1 \\ v_1 \end{pmatrix} + B_{t_1-t_0}(r) \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \end{pmatrix}$$

with

$$A = \mathcal{D}_{t_1-t_0} W(r) W(1)^{-1} \mathcal{D}_{t_1-t_0}^{-1}$$

$$B = \mathcal{E}_{t_1-t_0}(r) - \mathcal{D}_{t_1-t_0} W(r) W(1)^{-1} \mathcal{D}_{t_1-t_0}^{-1} \mathcal{E}_{t_1-t_0}(1)$$

Which g_1, g_2?

Best: $g_1, g_2 \sim r^{3/2}$ but then $W(r)$ not invertible

$g_1 \sim r^{\frac{3}{2}+\varepsilon_1}$, $g_2 \sim r^{\frac{3}{2}+\varepsilon_2}$ better but not good enough.

$g_1(r) = r^{\frac{3}{2}} \cos(\log r)$, $g_2(r) = r^{\frac{3}{2}} \sin(\log r)$

(reminiscent of $r^{\frac{3}{2} \pm i}$ as $r^i = e^{i \log r} = \cos \log r + i \sin \log r$)

1) $\det A(r) = r^{2d} = \left(r^{\frac{1}{2}}\right)^{(3d+d)}$ (homogeneous dimension)

2) $\left|\left(A(r)^{-1}\right)_{i;2}\right| \lesssim (1 + |t_1 - t_0|)\, r^{-\frac{1}{2}}$ $i = 1,2$ $r \in (0,1]$.

3) ...

Balancing (1) & (2) $\hat{=}$ critical Σ

crucial in order for $(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot S$ regularity.

$\Sigma \nabla_v f$