Kinetic maximal L^p-regularity

and applications to quasilinear equations

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The problem

We study L^p-solutions $u = u(t,x,v)\colon [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ to linear kinetic equations of the type

$$\begin{cases} \partial_t u + v\cdot\nabla_x u = Au + f \\ u(0) = g, \end{cases} \tag{1}$$

where A is an operator in a suitable function space X and f, g are given data.
Goal: Characterize unique solutions u to equation (1) with $\partial_t u + v\cdot\nabla_x u, Au \in L^p((0,T);X)$ in terms of functions spaces for the data f and g. In particular, show that the solutions to equation (1) define a semi-flow in the trace space.

If A admits such a characterization we say that A enjoys **kinetic maximal $L^p(X)$-regularity**.

Divide and conquer

- The case of **vanishing initial data**, i.e. $g = 0$. Using singular integral theory and the solution representation given by a fundamental solution. Complicated operators can often be reduced to simpler cases.
- The **homogeneous** case, i.e. $f = 0$. Make sense of the temporal trace. How does the kinetic term transfer regularity from v to x on this level?

The most important example

For $\beta \in (0,2]$ consider the **(fractional) Kolmogorov equation** in $\mathbb{R}^{2n}$,

$$\begin{cases} \partial_t u + v\cdot\nabla_x u = -(-\Delta_v)^{\frac{\beta}{2}} u + f \\ u(0) = g. \end{cases}$$

The equation dictates that for strong L^p-solutions the right solution space is

$$\mathbb{E}(0,T) = \{u\colon u, \partial_t u + v\cdot\nabla_x u, (-\Delta_v)^{\frac{\beta}{2}} u \in L^p((0,T);L^p(\mathbb{R}^{2n}))\},$$

i.e. $X = L^p(\mathbb{R}^{2n})$. Using operator theoretic properties of the characteristics $(t,x,v)\mapsto(t,x+tv,v)$ we can prove

$$\mathbb{E}(0,T) \hookrightarrow C([0,T];L^p(\mathbb{R}^{2n})),$$

whence X_γ, the trace space of $\mathbb{E}(0,T)$, is well-defined and $\mathbb{E}(0,T) \hookrightarrow C([0,T];X_\gamma)$. A theorem of Bouchut (2002) shows that

$$\mathbb{E}(0,T) = \mathbb{E}(0,T) \cap L^p((0,T);H^{\frac{\beta}{\beta+1},p}(\mathbb{R}^{2n})).$$

Hence, the trace space should also have some regularity in x. Indeed, using methods from harmonic analysis and the fundamental solution, we prove

$$X_\gamma \cong B^{\beta(1-1/p)}_{pp,v}(\mathbb{R}^{2n}) \cap B^{\frac{\beta}{\beta+1}(1-1/p)}_{pp,x}(\mathbb{R}^{2n}).$$

In particular, $A = -(-\Delta_v)^{\beta/2}$ admits kinetic maximal L^p-regularity in $L^p(\mathbb{R}^{2n})$.

Extensions

- Lebesgue spaces with temporal weights of the form $t^{1-\mu}$ for $\mu \in (1/p,1]$. This allows to consider initial values with lower regularity and to see that solutions regularize instantaneously.
- Different exponents of integrability, p in time and q in space.
- For $q = 2$ we characterize weak solutions to the (fractional) Kolmogorov equation, cf. [1].
- Weights $(1+|x-tv|^2)^{j/2}$ and $(1+|v|^2)^{k/2}$ for $j,k\in\mathbb{R}$.

More examples

The linearization of many nonlinear kinetic models leads to the **(fractional) Kolmogorov equation with variable coefficients**.
The characterization of strong L^p-solutions for the Kolmogorov equation can be extended to

$$\begin{cases} \partial_t u + v\cdot\nabla_x u = a(t,x,v)\colon \nabla_v^2 u + b\cdot\nabla_v u + cu + f \\ u(0) = g \end{cases}$$

provided that $a(t,x,v) \geq \lambda > 0$, $b \in L^\infty$, $c \in L^\infty$ and that

- $(t,x,v)\mapsto a(t,x,v)$ is bounded and uniformly continuous **or**
- $(t,x,v)\mapsto a(t,x+tv,v)$ is bounded and uniformly continuous, see [2].

Moreover, we can treat the case when a is not uniformly elliptic.
In [3] we study **non-local** operators with variable coefficients

$$[A^a_{t,x,v}u](t,x,v) = \text{p.v.}\int_{\mathbb{R}^n} (u(t,x,v+h) - u(t,x,v))\frac{a(t,x,v,h)}{|h|^{n+\beta}}\mathrm{d}h,$$

where a is symmetric in h, satisfies a similar continuity property in (t,x,v) and is also Hölder continuous in v.

An application

The precise solution theory allows to study **quasilinear kinetic partial differential equations** of the type

$$\begin{cases} \partial_t u + v\cdot\nabla_x u = A(u)u + F(u), t > 0 \\ u(0) = u_0. \end{cases}$$

We are interested in $L^p(X)$ solutions, that is functions such that $\partial_t u + v\cdot\nabla_x u, A(u)u \in L^p(X)$. Under a local Lipschitz assumption on the operators A and F we are able to prove local in time existence for all $u_0 \in X_\gamma$ such that $A(u_0)$ admits kinetic maximal $L^p(X)$-regularity.

Two Examples:

- A quasilinear diffusion problem, $A(w)u = \nabla_v\cdot(\kappa(w)\nabla_v u)$ for suitable functions κ.
- The kinetic toy model $A(w)u = M(w)\Delta_v u$, where $M(w)(t,x) = \int_{\mathbb{R}^n} u(t,x,v)\mathrm{d}v$ is the local mass.

The precise characterization of the trace space is required to control the non-linearities. For example, embeddings such as

$$B^{2-2/p}_{pp,v}(\mathbb{R}^{2n}) \cap B^{\frac{2}{3}(1-1/p)}_{pp,x}(\mathbb{R}^{2n}) \hookrightarrow C_0(\mathbb{R}^{2n})$$

are available for $p > 2n+1$.

Ongoing research

- Establish the kinetic maximal L^p-regularity for more operators, such as for example the kinetic Fokker-Planck equation $Au = \Delta_v u + v\cdot\nabla_v u$.
- Weak L^p-solutions.
- Study the local existence of solutions for more complicated quasilinear equations, e.g. the Landau equation.
- Global in time existence for quasilinear equations. Here, one needs to incorporate a priori estimates from the kinetic De Girogi-Nash-Moser theory.

References

[1] L. N. and R. Z. Kinetic maximal L^2-regularity for the (fractional) Kolmogorov equation. *Journal of Evolution Equations*, **21**, *2021*.

[2] L. N. and R. Z. Kinetic maximal L^p-regularity with temporal weights and application to quasilinear kinetic diffusion equations. *Journal of Differential Equations*, **307**, *2021*.

[3] L. N. Kinetic maximal $L^p_\mu(L^p)$-regularity for the fractional Kolmogorov equation with variable density. *Nonlinear Analysis*, **214**, *2022*.