Steady bubbles and drops

in inviscid fluids

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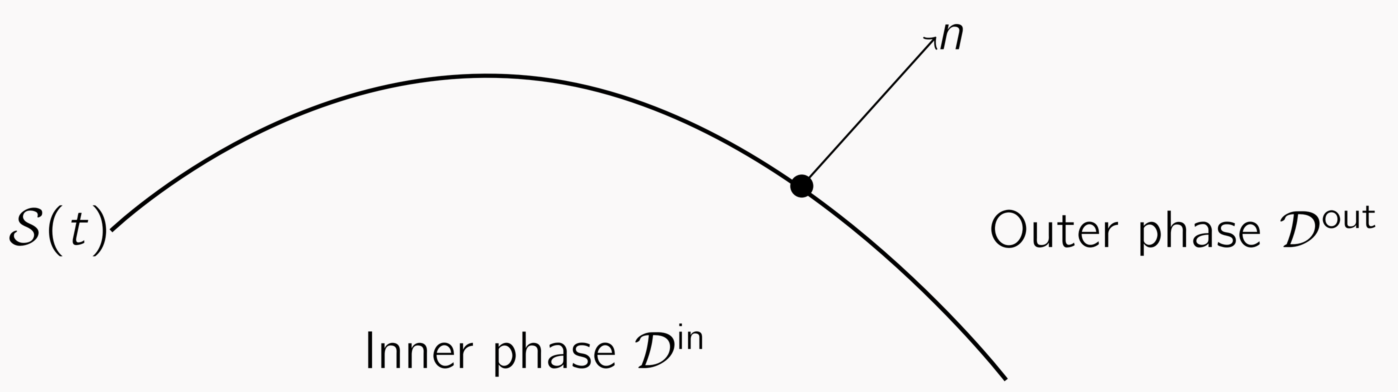
Two-phase Euler equations with surface tension

Velocity field of the fluid $U\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ solution to

$$\begin{aligned} \rho(\partial_t U + (U\cdot\nabla)U) + \nabla P &= 0 && \text{in } \mathbb{R}\times\mathbb{R}^3\\ \nabla\cdot U &= 0 && \text{in } \mathbb{R}\times\mathbb{R}^3\\ [\![P]\!] &= \sigma H && \text{on } \mathcal{S}(t)\\ [\![U\cdot n]\!] &= 0 && \text{on } \mathcal{S}(t)\end{aligned}$$

pressure P; surface tension $\sigma>0$; mean curvature H; jump $[\![f]\!] = f^{\mathrm{out}} - f^{\mathrm{in}}$
density $\rho(t) = \rho^{\mathrm{in}}\mathbb{1}_{\mathcal{D}^{\mathrm{in}}(t)} + \rho^{\mathrm{out}}\mathbb{1}_{\mathcal{D}^{\mathrm{out}}(t)}$ for $\rho^{\mathrm{in}},\rho^{\mathrm{out}} \geq 0$



Traveling wave solutions

For a given speed $V\geq 0$ we make the ansatz $\mathcal{S}(t) = \mathcal{S} + tVe_3$ and

$$u(x) = U(t,x_1,x_2,x_3+Vt) - Ve_3 \qquad p(x) = P(t,x_1,x_2,x_3+Vt).$$

The time-independent $u,p,\mathcal{S}$ solve the steady two-phase Euler equations with velocity field approaching $-Ve_3$ at infinity .
Bernoulli equations (for steady flows) for the inner/outer phase allow to rewrite the interfacial condition as

$$\frac{1}{2}[\![\rho|u|^2]\!] + \sigma H = \text{const} \quad \text{on } \mathcal{S}.$$

Assumptions:
Axisymmetric and swirl-free flow: $u = u(r,z)$ and azimutal $u_\varphi = 0$.
Uniform vorticity distribution in the inner phase for some $a\in\mathbb{R}$:

$$\operatorname{curl} u^{\mathrm{in}} = \omega_a = \frac{15}{2}a\begin{pmatrix}-x_2\\ x_1\\ 0\end{pmatrix} = \frac{15}{2}a\,re_\varphi.$$

Irrotational flow $\operatorname{curl} u^{\mathrm{out}} = 0$ in the outer domain.
Volume $\left|\mathcal{D}^{\mathrm{in}}\right| = \frac{4}{3}\pi R^3$.

Weber number: $\mathrm{We} = \dfrac{\rho^{\mathrm{out}}V^2R}{\sigma}$, **Vortex Weber number:** $\gamma = \dfrac{\rho^{\mathrm{in}}a^2R^5}{\sigma}$

Overdetermined free boundary value problem

Vector stream function $\psi\colon\mathbb{R}^3\to\mathbb{R}^3$ with $u = \operatorname{curl}\psi - Ve_3$. Decompose

$$\psi = \left(a\psi^{\mathrm{in}} + V/2\, s\sin\theta\, e_\varphi\right)\mathbb{1}_{\mathcal{D}^{\mathrm{in}}} + V\psi^{\mathrm{out}}\mathbb{1}_{\mathcal{D}^{\mathrm{out}}}$$

with $\psi^{\mathrm{in}}\colon\mathcal{D}^{\mathrm{in}}\to\mathbb{R}^3$ solution to

$$\begin{cases}-\Delta\psi^{\mathrm{in}} = \frac{15}{2}s\sin\theta\, e_\varphi & \text{in } \mathcal{D}^{\mathrm{in}},\\ \psi^{\mathrm{in}} = 0 & \text{on } \mathcal{S},\end{cases}$$

and $\psi^{\mathrm{out}}\colon\mathcal{D}^{\mathrm{out}}\to\mathbb{R}^3$ vanishing at infinity and solving

$$\begin{cases}-\Delta\psi^{\mathrm{out}} = 0 & \text{in } \mathcal{D}^{\mathrm{out}},\\ \psi^{\mathrm{out}} = \frac{1}{2}s\sin\theta\, e_\varphi & \text{on } \mathcal{S}.\end{cases}$$

Jump equation: $\dfrac{\gamma}{2}|\operatorname{curl}\psi^{\mathrm{in}}|^2 - \dfrac{\mathrm{We}}{2}|\operatorname{curl}\psi^{\mathrm{out}} - e_3|^2 + H = \text{const. on } \mathcal{S}.$

Hill's spherical vortex

A first solution is given by $\mathcal{S}$ the sphere of radius R, $V_S = |a|\,R^2\sqrt{\frac{\rho^{\mathrm{in}}}{\rho^{\mathrm{out}}}}$,

$$\psi_S(x) = \begin{pmatrix}-x_2\\ x_1\\ 0\end{pmatrix}\cdot\begin{cases}\dfrac{3a}{4}\left(R^2 - |x|^2\right) + \dfrac{V_S}{2} & \text{for } |x|\leq R\\ \dfrac{V_S}{2}\dfrac{R^3}{|x|^3} & \text{for } |x|>R,\end{cases}$$

Vortex sheet at $\mathcal{S}$ (jump of tangential velocity), whenever $V_S\neq aR^2$. Moreover,

$$\frac{1}{2}[\![\rho|\operatorname{curl}\psi_S - V_Se_3|^2]\!] = \frac{9}{8R^2}\left(a^2R^4\rho^{\mathrm{in}} - \rho^{\mathrm{out}}V_S^2\right)(x_1^2+x_2^2) = 0.$$

Perturbative ansatz

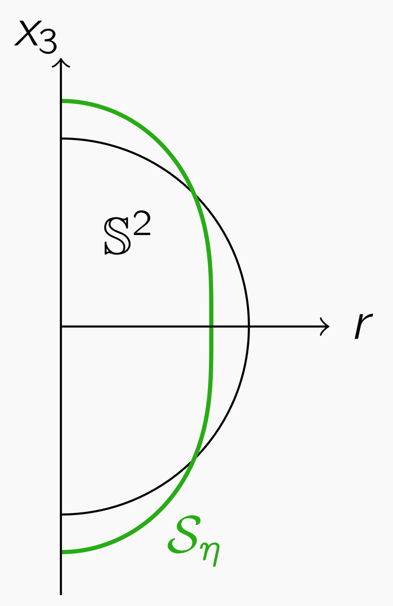
For a shape function $\eta\in\mathrm{H}^\beta(\mathbb{S}^2)$ with norm bounded by c_0 we consider

$$\mathcal{S}_\eta = \left\{(1+\eta(x))x : x\in\mathbb{S}^2\right\}.$$

We impose

1. axi-symmetry $\eta = \eta(\theta)$,
2. reflection invariance w.r.t. x_1x_2-plane,
3. the volume constraint $\left|\mathcal{D}^{\mathrm{in}}_\eta\right| = \frac{4}{3}\pi$,

and write $\eta\in\mathcal{M}^\beta_{c_0}$, $\chi_\eta = (1+\eta(x))x$.



Functional: $\mathcal{F}\colon\mathbb{R}\times\mathbb{R}\times\mathcal{M}^\beta_{c_0}\to\mathrm{H}^{\beta-2}_{\mathrm{sym}}(\mathbb{S}^2)/_{\mathrm{const}}$ defined as

$$\mathcal{F}(\gamma,\mathrm{We},\eta) = \frac{\gamma}{2}\left|(\operatorname{curl}\psi^{\mathrm{in}}_\eta)\circ\chi_\eta\right|^2 - \frac{\mathrm{We}}{2}\left|(\operatorname{curl}\psi^{\mathrm{out}}_\eta)\circ\chi_\eta - e_3\right|^2 + H_\eta\circ\chi_\eta.$$

Goal: find We, γ and η such that $\mathcal{F}(\gamma,\mathrm{We},\eta) = \text{const}$.

Theorem

Let $\beta>2$. There exists $c_0(\beta)>0$ and an increasing sequence $\Gamma = (\gamma_k)_{k\in\mathbb{N}}$ diverging to infinity as $k\to\infty$, and $\gamma_1\geq 1.862$, with the following property:

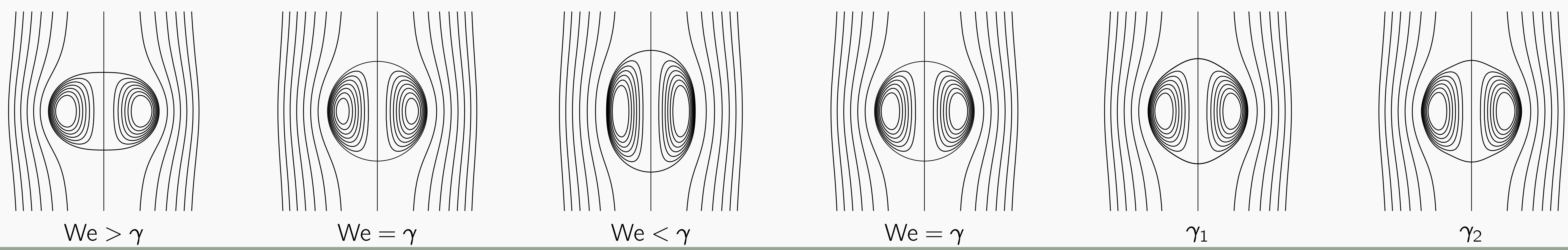
(1) For any $\gamma\in[0,\infty)\setminus\Gamma$ and any We close to but different from γ, there exists a unique nontrivial solution $\eta = \eta(\gamma,\mathrm{We})\in\mathcal{M}^\beta_{c_0}$ to the jump equation. This solution is smooth.

(2) For any $k\in\mathbb{N}$, there exists a unique local curve $s\mapsto\gamma(s)$ passing through γ_k and nontrivial smooth shape functions $\eta(s)\in\mathcal{M}^\beta_{c_0}$ such that the jump equation is solved with Weber numbers $(\gamma(s),\gamma(s))$.

Corollary: The spherical vortex is non-unique for $\mathrm{We} = \gamma\approx\gamma_k$.

Proof

1. Hill's spherical vortex: $\mathcal{F}(\gamma,\gamma,0) = 2 = \text{const}$.
2. **Linearisation:**
$$\langle \mathrm{D}_\eta\mathcal{F}(\gamma,\gamma,\eta)|_{\eta=0},\delta\eta\rangle = \frac{9}{2}\gamma\sin\theta\, e_\varphi\cdot(2\mathrm{Id}-\Lambda)(\sin\theta\,\delta\eta\, e_\varphi) - (\Delta_{\mathbb{S}^2} + 2\mathrm{Id})\,\delta\eta,$$
where Λ is the Dirichlet-to-Neumann map on the unit ball in $\mathbb{R}^3$.
3. Via spherical harmonics we reduce the analysis to the spectral properties of an infinite matrix operator in weighted sequence spaces $\rightsquigarrow\Gamma$.
4. At $\gamma\in[0,\infty)\setminus\Gamma$ we use the implicit function theorem $\rightsquigarrow$ (1).
5. At $\gamma\in\Gamma$ we employ the Crandall-Rabinowitz bifurcation theorem $\rightsquigarrow$ (2).



↑ Pictures and References ↓

[1] David Meyer, Lukas Niebel, and Christian Seis. Steady bubbles and drops in inviscid fluids, 2025. arXiv:2503.05503 to appear in Calc. Var. Partial Differential Equations.
[2] David Meyer and Christian Seis. Steady ring-shaped vortex sheets, 2024. arXiv:2409.08220.