# Kinetic maximal $L^p$-regularity

## and applications to quasilinear equations

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## The abstract problem

Given an operator $A$ acting on a suitable function space $X$ we are interested in solutions $u = u(t,x,v)\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of the kinetic Cauchy problem

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = Au + f \\ u(0) = g, \end{cases} \tag{1}$$

where $f, g$ are given data. For $L^p(X)$-solutions with $p \in (1,\infty)$, the natural solution space is

$$Z := \{u : \ u, \partial_t u + v \cdot \nabla_x u, \ Au \in L^p((0,T);X)\}.$$

**Goal 1:** Show the existence of a unique solution $u \in Z$ to equation (1) with $g = 0$ if $f \in L^p((0,T);X)$.
If $A$ has this property we say that $A$ admits **kinetic maximal $L^p(X)$-regularity**.
**Goal 2:** Investigate the temporal trace of functions in $Z$ and characterise the trace space in terms of accessible function spaces. This yields solvability of the Cauchy problem with nonzero initial value.

## The most important example

For $\beta \in (0,2]$ consider the **(fractional) Kolmogorov equation** in $\mathbb{R}^{2n}$,

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\frac{\beta}{2}} u + f \\ u(0) = g. \end{cases}$$

For strong $L^p$-solutions we choose $X = L^p(\mathbb{R}^{2n})$ and the solution space is

$$Z = \{u : \ u, \partial_t u + v \cdot \nabla_x u, \ (-\Delta_v)^{\frac{\beta}{2}} u \in L^p((0,T);L^p(\mathbb{R}^{2n}))\}.$$

The operator $A = -(-\Delta_v)^{\beta/2}$ admits **kinetic maximal $L^p(L^p(\mathbb{R}^{2n}))$-regularity** (Folland 1975, Chen/Zhang 2016).
**The trace space.** Using operator theoretic properties of the characteristics $(t,x,v) \mapsto (t, x - tv, v)$ we prove

$$Z \hookrightarrow C([0,T];L^p(\mathbb{R}^{2n})).$$

Hence $X_\gamma := \{g : \exists u \in Z \text{ with } u(0) = g\}$, the trace space of $Z$, is well-defined and $Z \hookrightarrow C([0,T];X_\gamma)$. A theorem of Bouchut (2002) on kinetic regularisation shows that

$$Z = Z \cap L^p((0,T);H_x^{\frac{\beta}{\beta+1},p}(\mathbb{R}^{2n})).$$

Hence, the trace space should also encode some regularity in $x$. Using the fundamental solution combined with methods from harmonic analysis, we prove

$$X_\gamma \cong B_{pp,x}^{\frac{\beta}{\beta+1}(1-1/p)}(\mathbb{R}^{2n}) \cap B_{pp,v}^{\beta(1-1/p)}(\mathbb{R}^{2n}).$$

## Extensions

- Kinetic maximal regularity in Lebesgue spaces with temporal weights of the form $t^{1-\mu}$ for $\mu \in (1/p, 1]$. This allows us to consider initial values with lower regularity and to see that solutions regularise instantaneously.

$$X_{\gamma,\mu} \cong B_{pp,x}^{\frac{\beta}{\beta+1}(\mu-1/p)}(\mathbb{R}^{2n}) \cap B_{pp,v}^{\beta(\mu-1/p)}(\mathbb{R}^{2n}) \ \text{ if } \ A = -(-\Delta_v)^{\beta/2}.$$

- Different exponents of integrability, $p$ in time and $q$ in space.
- For $q = 2$ we characterize weak solutions to the (fractional) Kolmogorov equation, cf. [1].
- Weights of the form $(1+|x-tv|^2)^{j/2}$, $(1+|x|^2+|v|^2)^{k/2}$ and $(1+|v|^2)^{l/2}$ for $j,k,l \in \mathbb{R}$.

## More examples

The linearization of many nonlinear kinetic equations leads to the **(fractional) Kolmogorov equation with variable coefficients**.
The characterisation of strong $L^p$-solutions for the Kolmogorov equation can be extended to

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = a(t,x,v) \colon \nabla_v^2 u + b \cdot \nabla_v u + cu + f \\ u(0) = g \end{cases}$$

provided that $0 < \lambda \le a(t,x,v) \le \Lambda$, $b \in L^\infty$, $c \in L^\infty$ and that

- $(t,x,v) \mapsto a(t,x,v)$ is uniformly continuous **or**
- $(t,x,v) \mapsto a(t,x+tv,v)$ is uniformly continuous, see [2].

Moreover, we can treat the case that $a$ is not uniformly elliptic.
In [3] we prove kinetic maximal $L^p$-regularity for **non-local** operators with variable coefficients

$$[Au](t,x,v) = \text{p.v.} \int_{\mathbb{R}^n} (u(t,x,v+h) - u(t,x,v)) \frac{a(t,x,v,h)}{|h|^{n+\beta}} \mathrm{d}h,$$

where $a$ is symmetric in $h$, satisfies a similar continuity property in $(t,x,v)$ and is also Hölder continuous in $v$.

## An application

The precise solution theory allows studying **quasilinear kinetic partial differential equations** of the type

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = A(u)u + F(u) \\ u(0) = u_0. \end{cases}$$

We are interested in $L^p(X)$ solutions, that is functions $u$ such that $\partial_t u + v \cdot \nabla_x u, A(u)u \in L^p(X)$. Under a local Lipschitz assumption on the mappings $A$ and $F$, we prove local (in time) existence for all $u_0 \in X_\gamma$ such that $A(u_0)$ admits kinetic maximal $L^p(X)$-regularity.
Here, the precise characterisation of the trace space is used to access embedding theorems, which allows controlling the nonlinearities.

**Examples:**

- A quasilinear diffusion problem, $A(w)u = \nabla_v \cdot (\kappa(w)\nabla_v u)$ for suitable functions $\kappa$.
- The Vlasov-Poisson-Fokker-Planck equation without friction

$$\begin{cases} \partial_t u + v \cdot \nabla_x u - \nabla_x E \cdot \nabla_v u = \sigma \Delta_v u \\ -\Delta_x E = \gamma \int_{\mathbb{R}^n} u(t,x,v) \mathrm{d}v, \end{cases}$$

where $\sigma > 0$ and $E = E(t,x)$ is the induced electric/gravitational ($\gamma = -1/\gamma = 1$) field.

## Ongoing research

- Establish kinetic maximal $L^p$-regularity for more operators, such as for example the kinetic Fokker-Planck equation $Au = \Delta_v u + v \cdot \nabla_v u$.
- Study weak $L^p$-solutions to the Kolmogorov equation.
- Study the local existence of solutions for more sophisticated quasilinear equations, e.g. the Landau equation.
- Global in time existence for quasilinear kinetic PDEs. Here, one needs to incorporate a priori estimates from the kinetic De Giorgi-Nash-Moser theory.

## References


[1] L. N. and R. Z. Kinetic maximal $L^2$-regularity for the (fractional) Kolmogorov equation. *Journal of Evolution Equations*, **21**, *2021*.

[2] L. N. and R. Z. Kinetic maximal $L^p$-regularity with temporal weights and application to quasilinear kinetic diffusion equations. *Journal of Differential Equations*, **307**, *2022*.

[3] L. N. Kinetic maximal $L^p_\mu(L^p)$-regularity for the fractional Kolmogorov equation with variable density. *Nonlinear Analysis*, **214**, *2022*.