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Trajectories and the De Giorgi-Nash-Moser theory

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Kinetic equations

Here: $(t, x, v) \in \Omega_T = (0, T) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2n}$.

Study particle density $f = f(t, x, v) \colon \Omega_T \to \mathbb{R}$

Kolmogorov equation

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Study particle density $f = f(t, x, v)\colon \Omega_T \to \mathbb{R}$ solution to

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \big(\mathfrak{a}(t, x, v) \nabla_v f\big) + S$$

Kolmogorov equation

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$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \big(\mathfrak{a}(t, x, v) \nabla_v f\big) + S$$

with $\mathfrak{a} \colon \Omega_T \to \mathbb{R}^{n \times n}$ measurable such that

(H1) $\lambda \, |\xi|^2 \leq \langle \mathfrak{a}(t, x, v)\xi, \xi \rangle$ for all $\xi \in \mathbb{R}^n$ and a.e. $(t, x, v) \in \Omega_T$

(H2) $\sum_{i,j=1}^{n} |\mathfrak{a}_{ij}(t, x, v)|^2 \leq \Lambda^2$ for a.e. $(t, x, v) \in \Omega_T$

and some constants $0 < \lambda < \Lambda$.
Moreover, $S \colon \Omega_T \to \mathbb{R}$ a source term.

Kolmogorov equation

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$f = f(t, x, v)\colon \Omega_T \to \mathbb{R}$ particle density solution to

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\mathfrak{a}(t, x, v)\nabla_v f)$$

with $\mathfrak{a}\colon \Omega_T \to \mathbb{R}^{n\times n}$ measurable, elliptic and bounded.

- Rough coefficients.
- Fokker-Planck equation to the integrated Wiener process.
- (Simplified) Version of the linearised Landau equation.
- For $\mathfrak{a} = \mathrm{Id}$ Kolmogorov constructed fundamental solution in 1934.

Kolmogorov equation

Consider $\mathfrak{a} = \mathrm{Id}$.

$$(\partial_t + v \cdot \nabla_x) f = \Delta_v f + S$$

Kolmogorov equation

Consider $\mathfrak{a} = \mathrm{Id}$.

Hörmander operator (type B) - hypoelliptic

$$(\partial_t + v \cdot \nabla_x) f = \sum_{i=1}^{n} \partial_{v_i}^2 f + S$$

Kolmogorov equation

Hörmander operator (type B) - hypoelliptic

$$X_0 f = \sum_{i=1}^{n} X_i^2 f + S$$

where $X_0 = \partial_t + v \cdot \nabla_x$ and $X_i = \partial_{v_i}$.

$$[\partial_{v_i}, \partial_t + v \cdot \nabla_x]f = \partial_{v_i}(\partial_t + v \cdot \nabla_x)f - (\partial_t + v \cdot \nabla_x)\,\partial_{v_i} f = \partial_{x_i} f$$

Kolmogorov equation

Hörmander operator (type B) - hypoelliptic

$$X_0 f = \sum_{i=1}^{n} X_i^2 f + S$$

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$$[\partial_{v_i}, \partial_t + v \cdot \nabla_x] f = \partial_{v_i}(\partial_t + v \cdot \nabla_x) f - (\partial_t + v \cdot \nabla_x)\, \partial_{v_i} f = \partial_{x_i} f$$

Theorem (Hörmander '67):
Assume rank condition. If $S \in C^\infty$, then $f \in C^\infty$.

Kinetic geometry

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f$$

Scaling invariance:

$\lambda \mapsto (\lambda^2 t, \lambda^3 x, \lambda v)$

Translation invariance:

$(t_0, x_0, v_0) \mapsto (t - t_0, x - x_0 - (t - t_0)v_0, v - v_0)$

Kinetic geometry

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f$$

Scaling invariance:

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Translation invariance:

$(t_0, x_0, v_0) \mapsto (t - t_0, x - x_0 - (t - t_0)v_0, v - v_0)$

Kinetic cylinders:

$Q_r(t_0, x_0, v_0)$
$= \{-r^2 < t - t_0 \leq 0,\ |x - x_0 - (t - t_0)v_0| < r^3,\ |v - v_0| < r^3\}$

Can work at unit scale from now on.

Energy estimate

Testing (1) with $f\varphi^2$ for a cutoff function φ yields (formally):

$$\sup_{t\in(-1,0]} \int_{B_1(0)} |f(t,\cdot)|^2 \,\mathrm{d}(x,v) + \int_{-1}^{0} \int_{B_1(0)} |\nabla_v f|^2 \,\mathrm{d}(t,x,v) \lesssim \int_{-2}^{0} \int_{B_2(0)} |f|^2 \,\mathrm{d}(t,x,v)$$

Natural solution space

$$L_t^\infty(L_{x,v}^2) \cap L_{t,x}^2(\dot{H}_v^1)$$

Weak solutions

Definition:

A function $f \in L^\infty_t L^2_{x,v}(\Omega_T) \cap L^2_{t,x} \dot{H}^1_v(\Omega_T)$ is a weak (sub-, super-) solution to (1) if for all $\varphi \in C_c^\infty(\Omega_T)$ with $\varphi \geq 0$ we have

$$\int_{(0,T)\times\mathbb{R}^{2n}} [-f(\partial_t + v \cdot \nabla_x)\varphi + \langle \mathfrak{a} \nabla_v f, \nabla_v \varphi \rangle] \, \mathrm{d}(t,x,v) = (\geq, \leq) \int_{(0,T)\times\mathbb{R}^{2n}} S \varphi \mathrm{d}(t,x,v).$$

Weak solutions

Definition:

A function $f \in L^\infty_t L^2_{x,v}(\Omega_T) \cap L^2_{t,x} \dot{H}^1_v(\Omega_T)$ is a weak (sub-, super-) solution to (1) if for all $\varphi \in C^\infty_c(\Omega_T)$ with $\varphi \geq 0$ we have

$$\int_{(0,T)\times\mathbb{R}^{2n}} [-f(\partial_t + v \cdot \nabla_x)\varphi + \langle \mathfrak{a} \nabla_v f, \nabla_v \varphi \rangle] \,\mathrm{d}(t,x,v) = (\geq, \leq) \int_{(0,T)\times\mathbb{R}^{2n}} S\varphi \mathrm{d}(t,x,v).$$

Literature:

- Regularity, existence and uniqueness of weak solutions together with P. Auscher and C. Imbert 24
- previous works: Carrillo 98, Albritton-Armstrong-Mourrat-Novack 24, N.-Zacher 21, Nyström-Litsgård 21

What can we say *a priori* about the

regularity of weak (sub-, super-) solutions?

De Giorgi-Nash-Moser theory

SULLA DIFFERENZIABILITÀ E L'ANALITICITÀ DELLE ESTREMALI DEGLI INTEGRALI MULTIPLI REGOLARI (*)

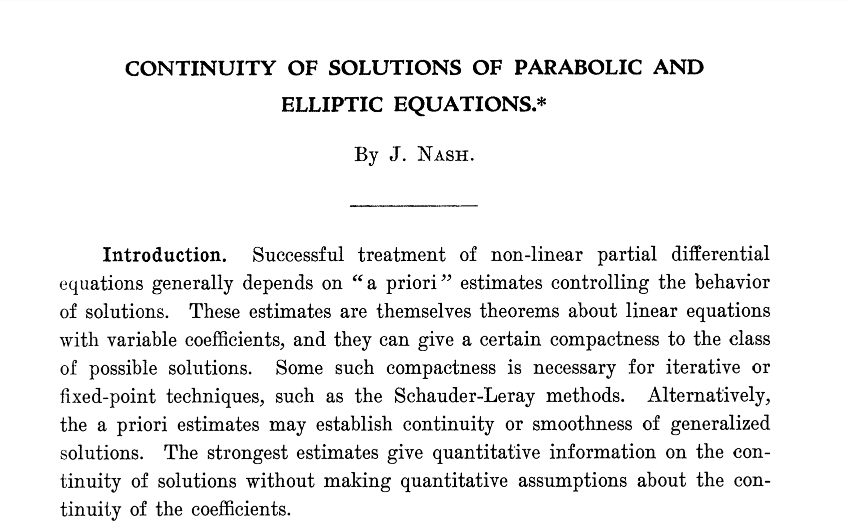
Memoria di ENNIO DE GIORGI
presentata dal Socio nazionale non residente Mauro PICONE
nell'adunanza del 24 Aprile 1957

Riassunto. — *Si studiano le estremali di alcuni integrali multipli regolari, supponendo nota a priori l'esistenza delle derivate parziali prime di quadrato sommabile; si dimostra il carattere hölderiano di tali derivate, da cui seguono l'indefinita differenziabilità e l'analiticità delle estremali.*

In questo lavoro mi occupo delle proprietà differenziali e specialmente dell'analiticità delle estremali degl'integrali multipli regolari; tale argomento è stato oggetto di molte ricerche da parte di matematici italiani e stranieri, sicchè appare assai difficile darne un quadro bibliografico completo; ci limiteremo quindi a citare qualche lavoro da cui il lettore potrà facilmente ricavare più ampie informazioni. Ricorderemo così i risultati di Hopf [3] (1), Stampacchia [9], Morrey [6], che danno teoremi di differenziabilità ed analiticità per estremali sempre meno regolari: precisamente si richiede l'esistenza di derivate seconde hölderiane in [3], di derivate prime hölderiane in [9], di derivate prime continue in [6]. A un di-

Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43.

De Giorgi-Nash-Moser theory


CONTINUITY OF SOLUTIONS OF PARABOLIC AND ELLIPTIC EQUATIONS.*

By J. Nash.

Introduction. Successful treatment of non-linear partial differential equations generally depends on "a priori" estimates controlling the behavior of solutions. These estimates are themselves theorems about linear equations with variable coefficients, and they can give a certain compactness to the class of possible solutions. Some such compactness is necessary for iterative or fixed-point techniques, such as the Schauder-Leray methods. Alternatively, the a priori estimates may establish continuity or smoothness of generalized solutions. The strongest estimates give quantitative information on the continuity of solutions without making quantitative assumptions about the continuity of the coefficients.

Amer. J. Math. **80** (1958), 931–954.

De Giorgi-Nash-Moser theory

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XIV, 577–591 (1961)

K. O. Friedrichs anniversary issue

On Harnack's Theorem for Elliptic Differential Equations*

JÜRGEN MOSER

1. Introduction

The theorem of Harnack referred to in the title is the following: If u is a positive harmonic function in a domain D, then in any compact set D' contained in D the inequality

$$\max_{D'} u \leqq c \min_{D'} u, \tag{1.1}$$

holds where the constant $c > 1$ depends on D and D' only. Equivalently, if

C.-G. Axel Harnack, Die Grundlagen der Theorie des logarithmischen Potentiales
und der eindeutigen Potentialfunktion in der Ebene, Teubner, Leipzig, 1887.

De Giorgi-Nash-Moser theory

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL XVII, 101–134 (1964)

A Harnack Inequality for Parabolic Differential Equations*

JÜRGEN MOSER

1. Introduction

(a) This paper is concerned with weak solutions of a parabolic differential equation

$$\frac{\partial u}{\partial t} = \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k} (a_{kl}(t, x)) \frac{\partial u}{\partial x_l} \tag{1.1}$$

with variable coefficients $a_{kl}(t, x)$. It is our aim to derive statements about the pointwise behavior of the solutions even if the coefficients are only measurable functions satisfying

Correction to: "A Harnack inequality for parabolic differential equations". Comm. Pure Appl. Math. **20** (1967), 231–236.

De Giorgi-Nash-Moser theory

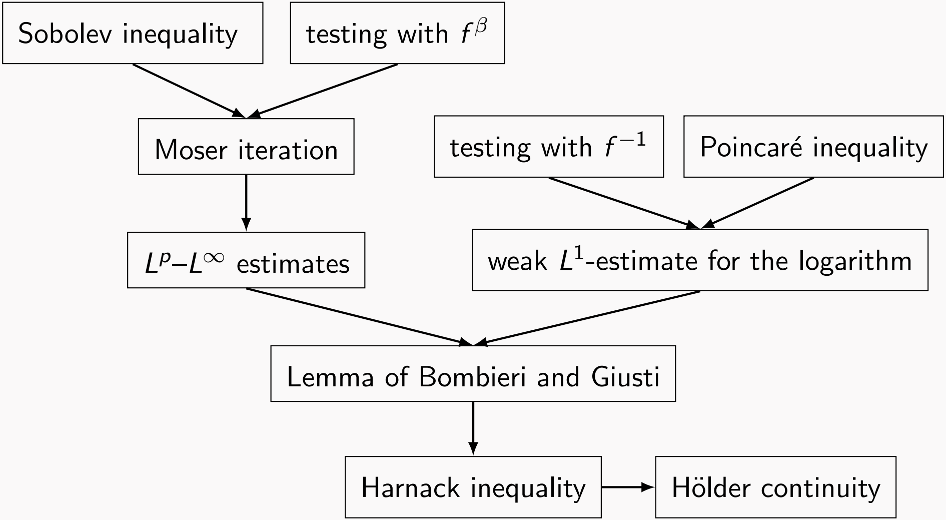
COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XXIV, 727–740 (1971)

On a Pointwise Estimate for Parabolic Differential Equations*

J. MOSER

§1. The purpose of this note is to describe a simplified proof of a theorem on linear parabolic differential equations which was published earlier in this journal (cf. [6]). This theorem gives a pointwise estimate for positive weak solutions of linear parabolic differential equations and is usually referred to as the Harnack inequality since it generalizes a classical inequality by Harnack for positive harmonic functions. The proof of this theorem for parabolic equations with variable coefficients uses a collection of *a priori* estimates for the powers and the logarithm of the solutions which are played out against each other with the help of general inequalities, primarily consequences of Sobolov's inequality. At one point, however, our previous argument required a new estimate (called Main Lemma in [6]) which generalizes an interesting theorem by F. John and L. Nirenberg. The proof of this lemma is quite intricate and it was desirable to avoid it entirely.

Moser's 1971 method



Goal:

Moser's 1971 method in kinetic theory

Kinetic De Giorgi-Nash-Moser theory

$L^p - L^\infty$-estimates

Communications in Contemporary Mathematics
Vol. 6, No. 3 (2004) 395–417

THE MOSER'S ITERATIVE METHOD FOR A CLASS OF ULTRAPARABOLIC EQUATIONS

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Revised 21 May 2003

We adapt the iterative scheme by Moser, to prove that the weak solutions to an ultraparabolic equation, with measurable coefficients, are locally bounded functions. Due to the strong degeneracy of the equation, our method differs from the classical one in that it is based on some ad hoc Sobolev type inequalities for solutions.

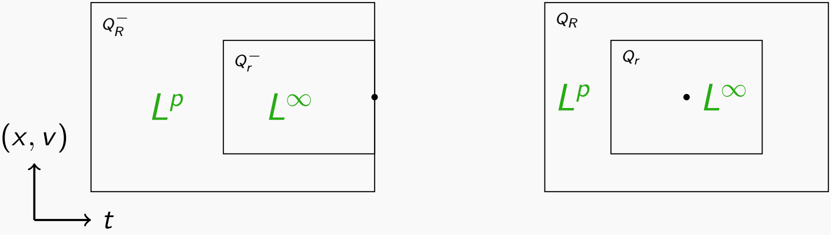
$L^p - L^\infty$ estimates

Theorem (Pascucci-Polidoro 04):

Let $\delta \in (0,1)$, $\delta \leq r < R \leq 1$. There exists $C(n, \lambda, \Lambda, \delta) > 0$ such that any positive weak solution f to (1) satisfies

$$\sup_{Q_r^-} f^p \leq \frac{c}{(R-r)^{4n+2}} \int_{Q_R^-} f^p \mathrm{d}(t,x,v) \qquad p < 0,$$

$$\sup_{Q_r} f^p \leq \frac{c}{(R-r)^{4n+2}} \int_{Q_R} f^p \mathrm{d}(t,x,v) \qquad p > 0.$$



Hölder continuity

Communications in Contemporary Mathematics
Vol. 13, No. 3 (2011) 375–387

DOI: 10.1142/S0219199711004385

THE C^α REGULARITY OF A CLASS OF ULTRAPARABOLIC EQUATIONS

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Received 18 September 2008
Revised 28 June 2009

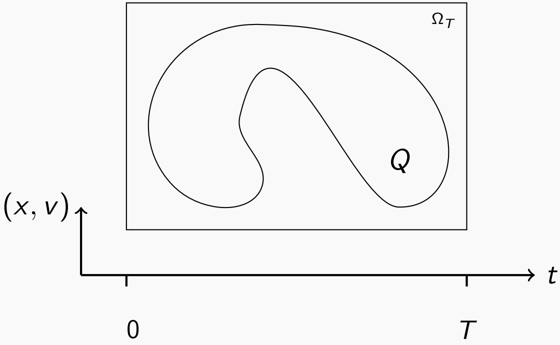
We prove the C^α regularity for weak solutions to a class of ultraparabolic equation, with measurable coefficients. The result generalized our recent C^α regularity results of Prandtl's system to high dimensional cases.

Hölder continuity

Theorem (Zhang 11, Wang & Zhang 09,11):

Let f be a weak solution to (1) and $Q \subset\subset \Omega_T$. Then, there exist constants $\varepsilon, C > 0$ such that $f \in \dot{C}^{\varepsilon}_{\mathrm{kin}}(\bar{Q})$ and

$$\|f\|_{\dot{C}^{\varepsilon}_{\mathrm{kin}}(\bar{Q})} \leq C \|f\|_{L^\infty(\Omega_T)}.$$



Harnack inequality

Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XIX (2019), 253-295

Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation

FRANÇOIS GOLSE, CYRIL IMBERT, CLÉMENT MOUHOT AND ALEXIS F. VASSEUR

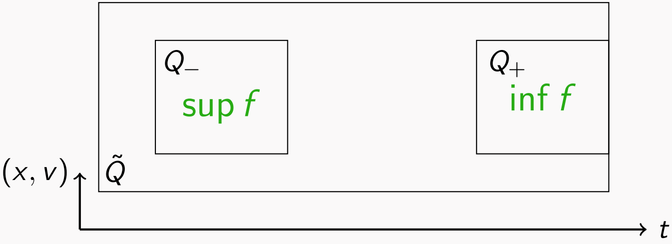
Abstract. We extend the De Giorgi-Nash-Moser theory to a class of kinetic Fokker-Planck equations and deduce new results on the Landau-Coulomb equation. More precisely, we first study the Hölder regularity and establish a Harnack inequality for solutions to a general linear equation of Fokker-Planck type whose coefficients are merely measurable and essentially bounded, *i.e.* assuming no regularity on the coefficients in order to later derive results for non-linear problems. This general equation has the formal structure of the hypoelliptic equations "of type II", sometimes also called ultraparabolic equations of Kolmogorov type,

Harnack inequality

Theorem (GIMV 19):

There exists a universal const $C = C(n, \lambda, \Lambda) > 0$ such that for any nonnegative weak solution f of (1) in $\tilde{Q}$ we have

$$\sup_{Q_-} f \leq C \inf_{Q_+} f.$$



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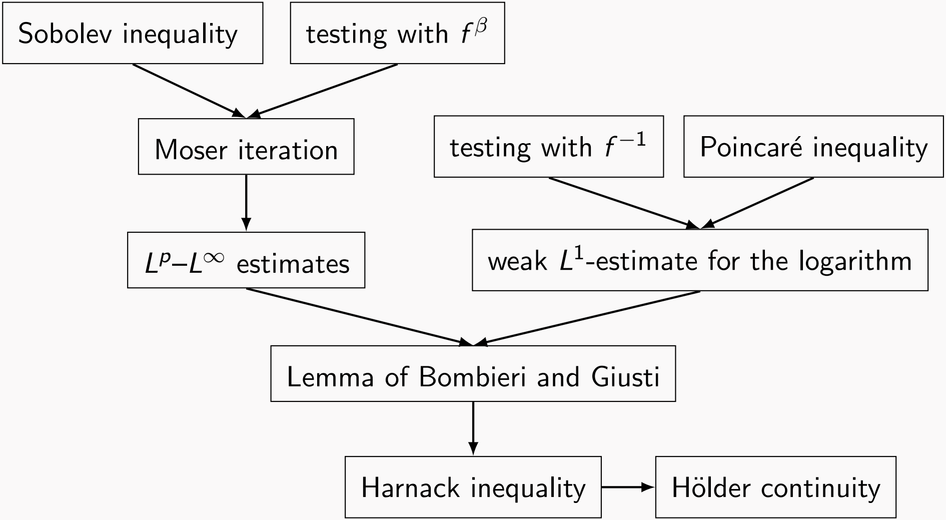
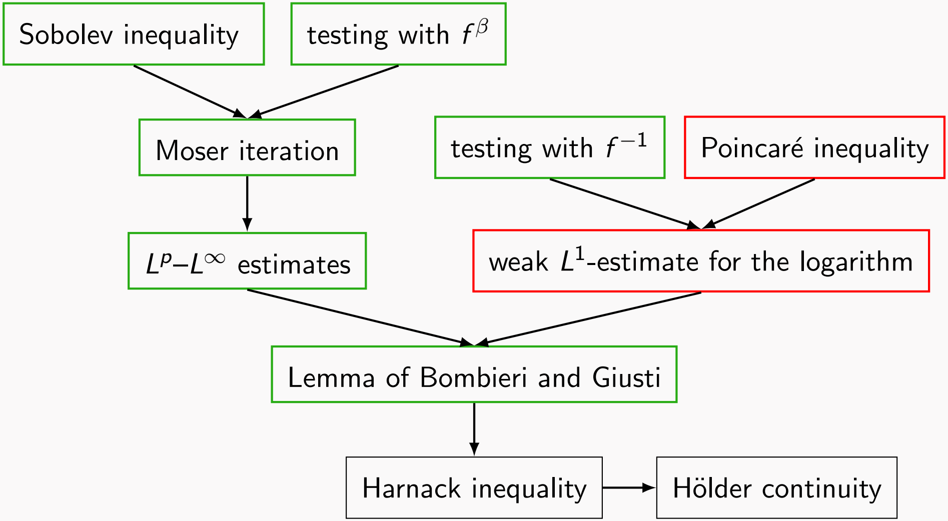
Towards

Moser's 1971 method in kinetic theory

Towards Moser's 1971 method in kinetic theory



Towards Moser's 1971 method in kinetic theory



The logarithm

Suppose that f is a positive weak supersolution to

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\mathfrak{a}(t, x, v) \nabla_v f)$$

then the $g = \log f$ is a weak supersolution to

$$\partial_t g + v \cdot \nabla_x g = \nabla_v \cdot (\mathfrak{a} \nabla_v g) + \langle \mathfrak{a} \nabla_v g, \nabla_v g \rangle.$$

Lemma of Bombieri and Giusti

Lemma (Moser 71, Bombieri and Giusti 72):

Let (X, ν) be a finite measure space, $U_\sigma \subset X$, $0 < \sigma \leq 1$ measurable with $U_{\sigma'} \subset U_\sigma$ if $\sigma' \leq \sigma$. Let $C_1, C_2 > 0$, $\delta \in (0, 1)$, $\tilde{\mu} > 1$, $\gamma > 0$. Suppose $0 \leq f \colon U_1 \to \mathbb{R}$ satisfies the following two conditions:

- for all $0 < \delta \leq r < R \leq 1$ and $0 < p < 1/\tilde{\mu}$ we have

$$\sup_{U_r} f^p \leq \frac{C_1}{(R-r)^\gamma \nu(U_1)} \int_{U_R} f^p \mathrm{d}\nu$$

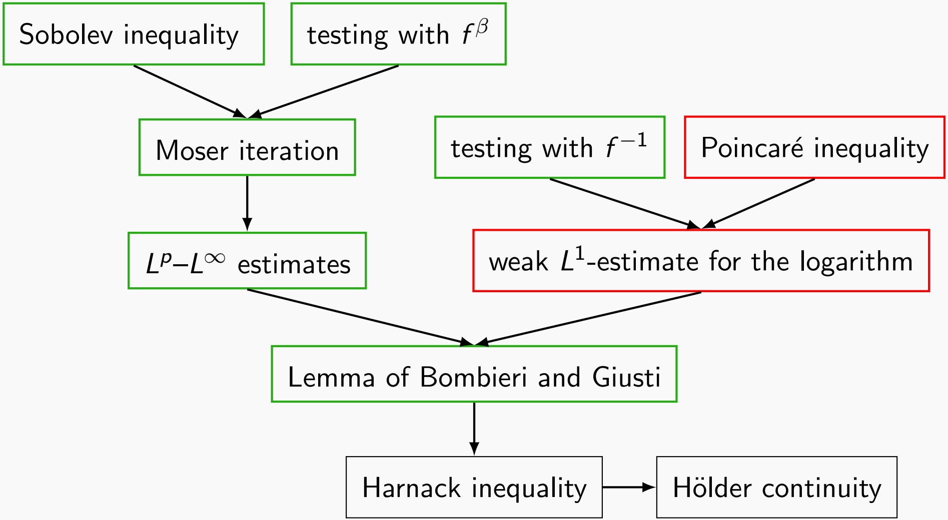
- $s\nu(\{\log f > s\}) \leq C_2 \tilde{\mu}\, \nu(U_1)$ for all $s > 0$.

Then

$$\sup_{U_\delta} f \leq C^{\tilde{\mu}},$$

where $C = C(C_1, C_2, \delta, \gamma)$.

Towards Moser's 1971 method in kinetic theory



Jerison's Poincaré inequality

Theorem (Jerison 86):

Let $X_1, \ldots, X_m$ be smooth vector fields satisfying Hörmanders rank condition. Then,

$$\int_{B_r} |f - f_{B_r}|^2 \, \mathrm{d} \leqslant Cr^2 \int_{B_r} \sum_{i=1}^{m} |X_i f|^2 \, \mathrm{d}x.$$

Here, B_r are balls with respect to a natural metric.

Jerison's Poincaré inequality - kinetic?

Theorem (Jerison 86):

We have

$$\int_{Q_r} |f - f_{Q_r}|^2 \,\mathrm{d}(t,x,v) \leqslant Cr^2 \int_{Q_r} |\partial_t f + v \cdot \nabla_x f|^2 + |\nabla_v f|^2 \,\mathrm{d}(t,x,v).$$

Here, Q_r are kinetic cylinders.

Jerison's Poincaré inequality - kinetic?

Theorem (Jerison 86):

We have

$$\int_{Q_r} |f - f_{Q_r}|^2 \,\mathrm{d}(t,x,v) \leqslant Cr^2 \int_{Q_r} |\partial_t f + v \cdot \nabla_x f|^2 + |\nabla_v f|^2 \,\mathrm{d}(t,x,v).$$

Here, Q_r are kinetic cylinders.

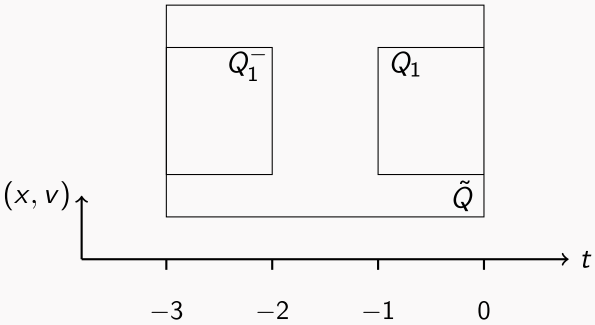
Need to treat $\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot h$, for some $h \in L^2$ at the correct scale.

Kinetic Poincaré inequality

Theorem (Guerand & Mouhot 22, N. & Zacher 22):

Let $g \in L^1(\tilde{Q}; \mathbb{R}^n)$ and φ^2 be supported in Q_1^-. Then, there exists a constant $C = C(n, \varphi) > 0$ such that for all subsolutions $f \geq 0$ to (1) in $\tilde{Q}$ we have

$$\left\| (f - \langle f \varphi^2 \rangle_{Q_1^-})_+ \right\|_{L^1(Q_1)} \leq C \left(\|\nabla_v f\|_{L^1(\tilde{Q})} + \|h\|_{L^1(\tilde{Q})} \right)$$



Kinetic Poincaré inequality

Theorem (Guerand & Mouhot 22, N. & Zacher 22):

Let $g \in L^1(\tilde{Q}; \mathbb{R}^n)$ and φ^2 be supported in Q_1^-. Then, there exists a constant $C = C(n, \varphi) > 0$ such that for all subsolutions $f \geq 0$ to (1) in $\tilde{Q}$ we have

$$\left\| (f - \langle f \varphi^2 \rangle_{Q_1^-})_+ \right\|_{L^1(Q_1)} \leq C \left(\|\nabla_v f\|_{L^1(\tilde{Q})} + \|h\|_{L^1(\tilde{Q})} \right)$$

Spacetime Poincaré inequalities are “too weak”.

Trajectories

Euclidean $f(v)$ - Poincaré inequality:

$$f(v) - f(w) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} f\left(w + r(v - w)\right) \mathrm{d}r$$

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Parabolic $f(t, v)$

$$f(t, v) - f(\eta, w) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} f(\gamma(r)) \mathrm{d}r$$

with $\gamma \colon [0, 1] \to \mathbb{R} \times \mathbb{R}^n$ with $\gamma(0) = (\eta, w)$ and $\gamma(1) = (t, v)$.

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Parabolic trajectory: $\gamma(r) = (\eta + r(t-\eta), w + r^{1/2}(v-w))$

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$$= \int_0^1 (t - \eta)[\partial_t f](\gamma(r)) + \frac{1}{2} r^{-\frac{1}{2}} (v - w) \cdot [\nabla f](\gamma(r)) \mathrm{d}r$$

with $\gamma \colon [0, 1] \to \mathbb{R} \times \mathbb{R}^n$ with $\gamma(0) = (\eta, w)$ and $\gamma(1) = (t, v)$.

Parabolic trajectory: $\gamma(r) = (\eta + r(t - \eta), w + r^{1/2}(v - w))$

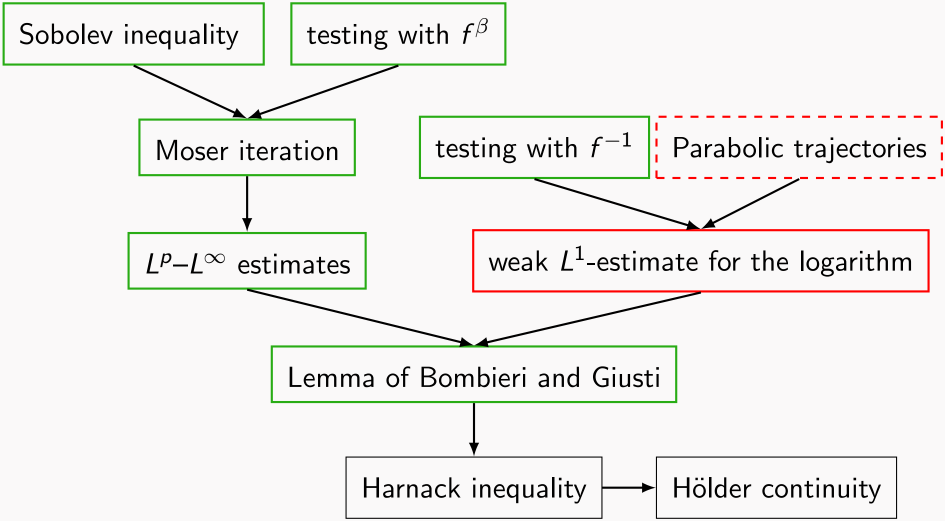
Moser's 1971 method and parabolic trajectories

A TRAJECTORIAL INTERPRETATION OF MOSER'S PROOF OF THE HARNACK INEQUALITY

LUKAS NIEBEL* AND RICO ZACHER

ABSTRACT. In 1971 Moser published a simplified version of his proof of the parabolic Harnack inequality. The core new ingredient is a fundamental lemma due to Bombieri and Giusti, which combines an $L^p - L^\infty$-estimate with a weak L^1-estimate for the logarithm of supersolutions. In this note, we give a novel proof of this weak L^1-estimate. The presented argument uses *parabolic trajectories* and does not use any Poincaré inequality. Moreover, the proposed argument gives a geometric interpretation of Moser's result and could allow transferring Moser's method to other equations.

Towards Moser's 1971 method in kinetic theory



Kinetic trajectories

Can we walk from (t, x, v) to (η, y, w) along $\partial_t + v \cdot \nabla_x$ and $\partial_{v_1}, \ldots \partial_{v_n}$?

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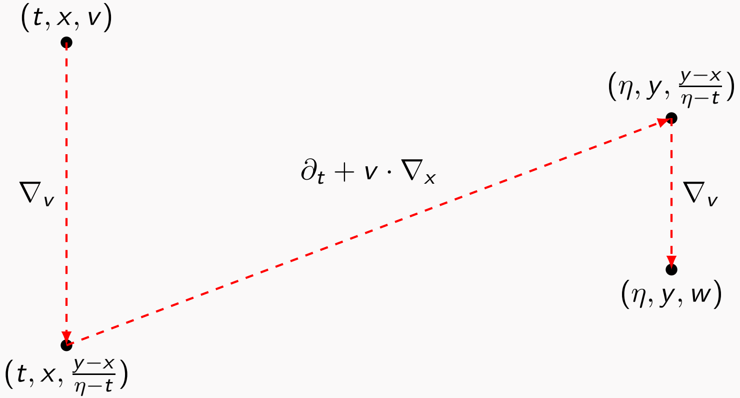

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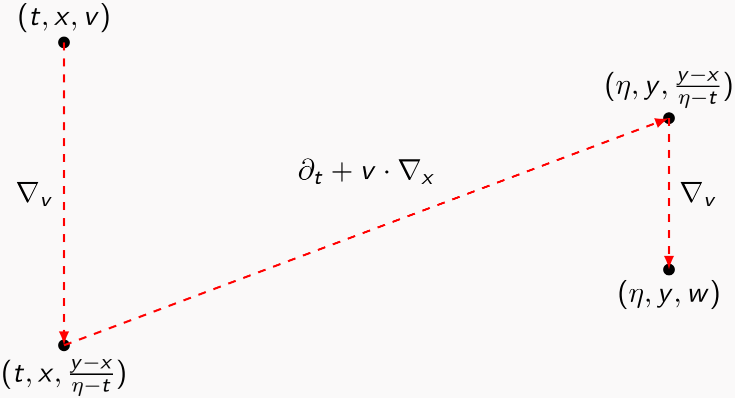
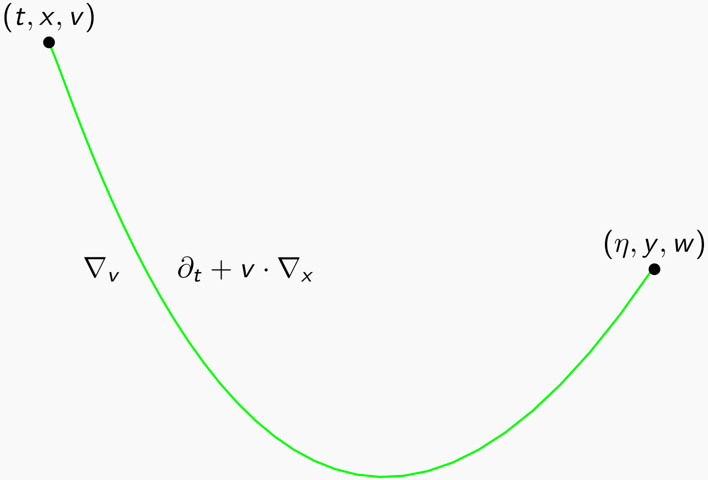


J. Guerand and C. Mouhot. *Quantitative De Giorgi methods in kinetic theory*, J. École polytech. - Math. **9** (2022), 1159-1181.

Kinetic trajectories

Can we walk from (t, x, v) to (η, y, w) along $\partial_t + v \cdot \nabla_x$ and $\partial_{v_1}, \dots \partial_{v_n}$?

L. N. and R. Zacher. *On a kinetic Poincaré inequality and beyond*, arXiv:2212.03199 (2022).



Kinetic trajectories

Definition:

Let (t, x, v) and $(\eta, y, w) \in \mathbb{R}^{1+2n}$ with $\eta \neq t$. A kinetic trajectory is a map

$$\gamma = \gamma(r) = \gamma(r; (t, x, v), (\eta, y, w)) = (\gamma_t(r), \gamma_x(r), \gamma_v(r)) \in \mathbb{R}^{1+2n}$$

defined for $r \in [0, 1]$ that is

- continuous on $r \in [0, 1]$ (and in particular bounded),
- differentiable on $r \in (0, 1)$,
- with endpoints $\gamma(0) = (t, x, v)$ and $\gamma(1) = (\eta, y, w)$,
- satisfying the constraint $\dot{\gamma}_x(r) = \dot{\gamma}_t(r)\gamma_v(r)$ for $r \in (0, 1)$.

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For $g \colon \mathbb{R}^{1+2n} \to \mathbb{R}$ smooth

$$\frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) = \dot{\gamma}_t(r)[\partial_t g] + \dot{\gamma}_x(r) \cdot [\nabla_x g](\gamma(r)) + \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r))$$

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For $g \colon \mathbb{R}^{1+2n} \to \mathbb{R}$ smooth

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) &= \dot{\gamma}_t(r)[\partial_t g] + \dot{\gamma}_x(r) \cdot [\nabla_x g](\gamma(r)) + \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r)) \\ &= \dot{\gamma}_t(r)[\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r)).\end{aligned}$$

Literature on trajectories

- Early works by Carathéodory 09, Rashevskii 38 and Chow 39.
- Breakthrough by Nagel, Stein and Wainger 85.
- Lots of works on Geometric Control theory.
- Trajectorial proof of Jerison's Poincaré inequality by Lanconelli-Morbidelli 00.
- Kinetic trajectories are constructed in Pascucci-Polidoro 04.

In none of these results X_0 and $X_1, \dots, X_n$ are treated at the right scale.

Critical kinetic trajectories

Today $\dot{\gamma}_t = \eta - t$.

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A kinetic trajectory is called a critical kinetic trajectory if it additionally satisfies

$$\left| \left(\nabla_{y,w} \gamma(r; (t,x,v), (\eta,y,w))^{-1} \right)_{\cdot;2} \right| \sim |\dot{\gamma}_v(r)| \sim r^{-\frac{1}{2}}$$

as $r \to 0$, $r \neq 0$.

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Trajectories constructed in N.-Zacher 22 are not critical.

Neither are the ones in the follow-up work:

F. Anceschi, H. Dietert, J. Guerand, A. Loher, C. Mouhot, and A. Rebucci. Poincaré inequality and quantitative De Giorgi method for hypoelliptic operators, 2024.

Critical kinetic trajectories

Lemma (DMNZ 24):

There exists a family of critical kinetic trajectories given by

$$\gamma(r) = \begin{pmatrix} \gamma_t(r) \\ \gamma_x(r) \\ \gamma_v(r) \end{pmatrix} = \begin{pmatrix} t + (\eta - t)r \\ \mathcal{A}_{\eta-t}(r) \begin{pmatrix} y \\ w \end{pmatrix} + \mathcal{B}_{\eta-t}(r) \begin{pmatrix} x \\ v \end{pmatrix} \end{pmatrix}$$

with properties such as

- $\mathcal{A}_{\eta-t}(0) = 0$, $\mathcal{A}_{\eta-t}(1) = \mathrm{Id}_{2n}$ and $\mathcal{B}_{\eta-t}(0) = \mathrm{Id}_{2n}$, $\mathcal{B}_{\eta-t}(1) = 0$
- $\det \mathcal{A}_{\eta-t}(r) = r^{2n}$, $\det \mathcal{B}_{\eta-t}(r) \approx (1-r)^{2n}$
- spatial uniform control $\gamma(r) \in \tilde{Q}$
- criticality, i.e. $|\dot{\gamma}_v| \lesssim r^{-\frac{1}{2}}$ and

$$\left| \left(\nabla_{y,w} \gamma(r; (t,x,v), (\eta,y,w))^{-1} \right)_{\cdot;2} \right| = \left| (\mathcal{A}^{-1}_{\eta-t})_{\cdot;2} \right| \lesssim r^{-\frac{1}{2}}.$$

Construction of kinetic trajectories

Ansatz:

$$\dot{\gamma}_t = \eta - t \text{ and } \dot{\gamma}_v = \ddot{g}_0(r)\mathsf{m}_0 + \ddot{g}_1(r)\mathsf{m}_1$$

for two forcings $\ddot{g}_0, \ddot{g}_1 \colon [0,1] \to \mathbb{R}$ and vectorial parameters $\mathsf{m}_0, \mathsf{m}_1 \in \mathbb{R}^n$.

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Integration yields

$$\begin{cases} \dot{\gamma}_v(r) = \ddot{g}_0(r)\mathbf{m}_0 + \ddot{g}_1(r)\mathbf{m}_1 \\ \gamma_v(r) = \dot{g}_0(r)\mathbf{m}_0 + \dot{g}_1(r)\mathbf{m}_1 + v. \end{cases}$$

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A kinetic trajectory needs to satisfy

$$\dot{\gamma}_x(r) = \dot{\gamma}_t(r)\gamma_v(r) = (\eta - t)\dot{g}_0(r)\mathbf{m}_0 + (\eta - t)\dot{g}_1(r)\mathbf{m}_1 + (\eta - t)v$$

Construction of kinetic trajectories

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Endpoint condition determines the vectorial parameters

$$\begin{cases} \gamma_x(1) = (\eta - t)g_0(1)\mathbf{m}_0 + (\eta - t)g_1(1)\mathbf{m}_1 + (\eta - t)v + x = y \\ \gamma_v(1) = \dot g_0(1)\mathbf{m}_0 + \dot g_1(1)\mathbf{m}_1 + v = w \end{cases}$$

Construction of kinetic trajectories

Ansatz:

$$\dot{\gamma}_t = \eta - t \text{ and } \dot{\gamma}_v = \ddot{g}_0(r)\mathrm{m}_0 + \ddot{g}_1(r)\mathrm{m}_1$$

for two forcings $\ddot{g}_0, \ddot{g}_1 \colon [0,1] \to \mathbb{R}$ and vectorial parameters $\mathrm{m}_0, \mathrm{m}_1 \in \mathbb{R}^n$.

Integration yields

$$\begin{cases} \gamma_v(r) = \dot{g}_0(r)\mathbf{m}_0 + \dot{g}_1(r)\mathbf{m}_1 + v \\ \gamma_x(r) = (\eta - t)g_0(r)\mathbf{m}_0 + (\eta - t)g_1(r)\mathbf{m}_1 + (\eta - t)rv + x \end{cases}$$

Endpoint condition determines the vectorial parameters

$$\begin{cases} (\eta - t)g_0(1)\mathbf{m}_0 + (\eta - t)g_1(1)\mathbf{m}_1 + (\eta - t)v + x = y \\ \dot{g}_0(1)\mathbf{m}_0 + \dot{g}_1(1)\mathbf{m}_1 + v = w \end{cases}$$

Criticality is achieved for a good choice of the forcing.

Weak L^1-estimate for $\log f$

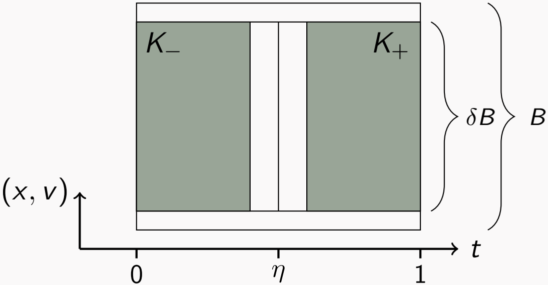
Theorem (DMNZ 24):

Let $\delta, \eta \in (0,1)$ and $\varepsilon > 0$. Then for any supersolution $f \geq \varepsilon > 0$ to (1) there exists a constant $C = C(n, \delta, \eta, \lambda, \Lambda) > 0$ such that

$$s\,|\{(t,x,v) \in K_- \colon \log f(t,x,v) - c(f) > s\}| \leq C$$

$$s\,|\{(t,x,v) \in K_+ \colon c(f) - \log f(t,x,v) > s\}| \leq C$$

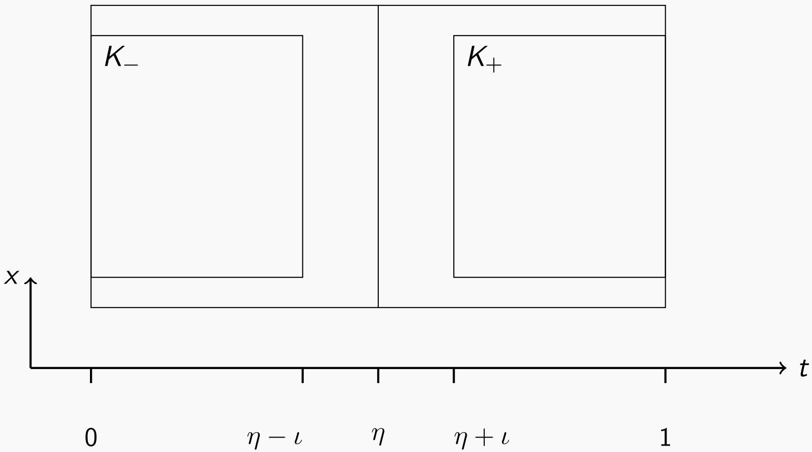
for all $s > 0$ with $c(f) = \dfrac{1}{c_\varphi} \displaystyle\int_B \log f(\eta, y, w) \varphi^2(y,w) \mathrm{d}(y,w)$.



Proof of the weak L^1-estimate

Unit size. $\mathfrak{a} = \mathrm{Id}$ for simplicity. Goal:

$$s\,|\{(t,x,v) \in K_- : \ \log f(t,x,v) - c(f) > s\}| \leq C, \quad s > 0$$



Proof of the weak L^1-estimate

Recall

$$c(f) = \frac{1}{c_\varphi} \int\limits_B [\log f](\eta, y, w)\varphi^2(y, w)\mathrm{d}(y, w).$$

where

$$c_\varphi = \int_B \varphi^2(y, w)\mathrm{d}(y, w).$$

Proof of the weak L^1-estimate

Recall

$$c(f) = \frac{1}{c_\varphi} \int\limits_B [\log f](\eta, y, w)\varphi^2(y, w)\mathrm{d}(y, w).$$

Note that

$$\begin{aligned} & s\,|\{(t, x, v) \in K_- \colon \log(f) - c(f) > s\}| \\ & \leq \int\limits_0^{\eta-\iota} \int\limits_B ([\log f](t, x, v) - c(f))_+ \mathrm{d}(t, x, v) \end{aligned}$$

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Goal: estimate

$$\int_0^{\eta-\iota} \int_B ([\log f](t, x, v) - c(f))_+ \mathrm{d}(t, x, v) \leq C$$

by a constant.

L^1-Poincaré inequality in spacetime without a gradient.

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$$\int_0^{\eta-\iota} \int_B ([\log f](t, x, v) - c(f))_+ \mathrm{d}(t, x, v) \le C$$

by a constant.

L^1-Poincaré inequality in spacetime without a gradient.

Recall: if f is supersolution to (1), then $g = \log f$ is a supersolution to

$$\partial_t g + v \cdot \nabla_x g = \Delta_v g + |\nabla_v g|^2$$

Proof of the L^1-estimate

For $g = \log f$ we have

$$g(t,x,v) - c(f)$$
$$= \frac{1}{c_\varphi} \int_B \left(g(t,x,v) - g(\eta,y,w)\right) \varphi^2(y,w) \mathrm{d}(y,w)$$
$$= -\frac{1}{c_\varphi} \int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) \mathrm{d}r \; \varphi^2(y,w) \mathrm{d}(y,w)$$

Proof of the L^1-estimate

For $g = \log f$ we have

$$
\begin{aligned}
&g(t,x,v) - c(f) \\
&= \frac{1}{c_\varphi} \int_B \left(g(t,x,v) - g(\eta,y,w)\right) \varphi^2(y,w)\mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r))\mathrm{d}r \ \varphi^2(y,w)\mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \dot\gamma_t(r)[\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot\gamma_v(r) \cdot [\nabla_v g](\gamma(r))\mathrm{d}r \ \varphi^2 \mathrm{d}(y,w)
\end{aligned}
$$

Proof of the L^1-estimate

For $g = \log f$ we have

$$
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&= -\frac{1}{c_\varphi} \int_B \int_0^1 \dot{\gamma}_t(r) [\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r)) \mathrm{d}r \; \varphi^2 \mathrm{d}(y,w) \\
&\leq -\frac{\eta - t}{c_\varphi} \int_B \int_0^1 [\Delta_v g](\gamma(r)) + |\nabla_v g|^2 (\gamma(r)) \mathrm{d}r \; \varphi^2(y,w) \mathrm{d}(y,w) \\
&\quad - \frac{1}{c_\varphi} \int_B \int_0^1 \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r)) \mathrm{d}r \; \varphi^2(y,w) \mathrm{d}(y,w)
\end{aligned}
$$

Proof of the L^1-estimate

$(1)\ \partial_t g + v \cdot \nabla_x g = \Delta_v g + |\nabla_v g|^2$

For $g = \log f$ we have

$$
\begin{aligned}
&g(t,x,v) - c(f) \\
&= \frac{1}{c_\varphi} \int_B \left(g(t,x,v) - g(\eta,y,w)\right) \varphi^2(y,w)\mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r))\mathrm{d}r\ \varphi^2(y,w)\mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \dot{\gamma}_t(r)[\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r))\mathrm{d}r\ \varphi^2\mathrm{d}(y,w) \\
&\le -\frac{\eta - t}{c_\varphi} \int_B \int_0^1 [\Delta_v g](\gamma(r)) + |\nabla_v g|^2 (\gamma(r))\mathrm{d}r\ \varphi^2(y,w)\mathrm{d}(y,w) \\
&\quad - \frac{1}{c_\varphi} \int_B \int_0^1 \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r))\mathrm{d}r\ \varphi^2(y,w)\mathrm{d}(y,w)
\end{aligned}
$$

Idea: use quadratic gradient term to absorb all gradients

The forcing terms

Recall that $|\dot{\gamma}_v| \lesssim r^{-\frac{1}{2}}$, hence

$$
\begin{aligned}
&- \frac{1}{c_\varphi} \int_B \int_0^1 \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r)) \mathrm{d}r \; \varphi^2(y,w) \mathrm{d}(y,w) \\
&\lesssim \int_B \int_0^1 r^{-\frac{1}{2}} \left| \nabla_v g \right| (\gamma(r)) \mathrm{d}r \; \varphi(y,w) \mathrm{d}(y,w)
\end{aligned}
$$

Partial integration

$$\int_B [\Delta_v g](\gamma(r))\varphi^2(y,w)\mathrm{d}(y,w)$$

Partial integration

Substitute $(\tilde{y}, \tilde{w}) = \Phi(y, w) = \Phi_{r,t,x,v,\eta}(y, w) := (\gamma_x(r), \gamma_v(r))$.

$$\int_B [\Delta_v g](\gamma(r))\varphi^2(y, w)\mathrm{d}(y, w)$$
$$= \int_{\Phi(B)} [\Delta_v g](\gamma_t(r), \tilde{y}, \tilde{w})\varphi^2(\Phi^{-1}(\tilde{y}, \tilde{w}))\,|\det \mathcal{A}|^{-1}\,\mathrm{d}(\tilde{y}, \tilde{w})$$

Partial integration

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$$= -\int_{\Phi(B)} \langle [\nabla_v g](\gamma_t(r), \tilde{y}, \tilde{w}), \nabla_v \varphi^2(\Phi^{-1}(\tilde{y}, \tilde{w}))\rangle\, |\det \mathcal{A}|^{-1}\, \mathrm{d}(\tilde{y}, \tilde{w})$$

$$= -2\int_{\Phi(B)} \langle [\nabla_v g](\gamma_t(r), \tilde{y}, \tilde{w}), [\nabla\varphi]^T(\Phi^{-1}(\tilde{y}, \tilde{w}))(\mathcal{A}(r)^{-1})_{\cdot 2}\rangle$$

$$\varphi(\Phi^{-1}(\tilde{y}, \tilde{w})) \cdot |\det \mathcal{A}|^{-1}\, \mathrm{d}(\tilde{y}, \tilde{w})$$

Partial integration

Substitute $(\tilde{y}, \tilde{w}) = \Phi(y, w) = \Phi_{r,t,x,v,\eta}(y, w) := (\gamma_x(r), \gamma_v(r))$.

$$
\begin{aligned}
&\int_B [\Delta_v g](\gamma(r))\varphi^2(y,w)\mathrm{d}(y,w) \\
&= \int_{\Phi(B)} [\Delta_v g](\gamma_t(r), \tilde{y}, \tilde{w})\varphi^2(\Phi^{-1}(\tilde{y},\tilde{w}))\,|\det \mathcal{A}|^{-1}\,\mathrm{d}(\tilde{y},\tilde{w}) \\
&= -\int_{\Phi(B)} \langle [\nabla_v g](\gamma_t(r), \tilde{y}, \tilde{w}), \nabla_v \varphi^2(\Phi^{-1}(\tilde{y},\tilde{w}))\rangle\,|\det \mathcal{A}|^{-1}\,\mathrm{d}(\tilde{y},\tilde{w}) \\
&= -2\int_{\Phi(B)} \langle [\nabla_v g](\gamma_t(r), \tilde{y}, \tilde{w}), [\nabla\varphi]^T(\Phi^{-1}(\tilde{y},\tilde{w}))(\mathcal{A}(r)^{-1})_{\cdot 2}\rangle \\
&\qquad\qquad \varphi(\Phi^{-1}(\tilde{y},\tilde{w}))\cdot|\det \mathcal{A}|^{-1}\,\mathrm{d}(\tilde{y},\tilde{w}) \\
&= -2\int_B \langle [\nabla_v g](\gamma(r)), [\nabla\varphi]^T(y,w)(\mathcal{A}(r)^{-1})_{\cdot 2}\rangle\varphi(y,w)\mathrm{d}(y,w) \\
&\lesssim r^{-1/2}\int_B |\nabla_v g|\,(\gamma(r))\varphi(y,w)\mathrm{d}(y,w),
\end{aligned}
$$

Distributing the good term

$$(g(t,x) - c(f))_+ \lesssim \int_0^1 \int_B \Big(M r^{-1/2} \left|\nabla_v g\right| (\gamma(r))\varphi(y,w) - \left|\nabla_v g\right|^2 (\gamma(r))\varphi^2(y,w) \Big)_+ \mathrm{d}(y,w)\mathrm{d}r$$

for some constant $M > 0$.

Integrating on K_-

$$\int_0^{\eta-\iota}\int_B (g(t,x,v)-c(f))_+ \mathrm{d}(t,x,v)$$
$$\leq \int_0^{\eta}\int_B\int_0^1\int_B \Big(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w) - \left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\Big)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)$$

Integrating on K_-

$$\int_0^{\eta-\iota}\int_B (g(t,x,v)-c(f))_+ \mathrm{d}(t,x,v)$$

$$\leq \int_0^{\eta}\int_B\int_0^1\int_B \left(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w)-\left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\right)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)$$

$$=\int_0^{\eta}\int_B\int_0^{\frac{1}{2}}\int_B \left(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w)-\left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\right)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)$$

$$+\int_0^{\eta}\int_B\int_{\frac{1}{2}}^1\int_B \left(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w)-\left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\right)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)$$

Integrating on K_-

$$
\begin{aligned}
&\int_0^{\eta-\iota}\int_B (g(t,x,v)-c(f))_+ \mathrm{d}(t,x,v)\\
&\le \int_0^{\eta}\int_B\int_0^1\int_B \Big(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w)-\left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\Big)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)\\
&=\int_0^{\eta}\int_B\int_0^{\frac12}\int_B \Big(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w)-\left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\Big)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)\\
&+\int_0^{\eta}\int_B\int_{\frac12}^1\int_B \Big(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w)-\left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\Big)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)\\
&\le\int_0^{\eta}\int_B\int_0^{\frac12}\int_B \Big(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w)-\left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\Big)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)\\
&+C =: I_1 + C
\end{aligned}
$$

for some $C > 0$ by Cauchy-Schwarz inequality.

I_1

Substitute $(\tilde{x}, \tilde{v}) = \Psi_{r,t,\eta,y,w}(x, v) := \gamma_{x,v}(r)$ and $\tilde{t} = t + r(\eta - t)$.

$$I_1 = \int_0^{\eta} \int_B \int_0^{\frac{1}{2}} \int_B \Big(M r^{-1/2} \left|\nabla_v g\right| (\gamma(r)) \varphi(y, w) - \left|\nabla_v g\right|^2 (\gamma(r)) \varphi^2(y, w) \Big)_+ \mathrm{d}(y, w) \mathrm{d}r \mathrm{d}(x, v) \mathrm{d}t$$

I_1

Substitute $(\tilde{x}, \tilde{v}) = \Psi_{r,t,\eta,y,w}(x, v) := \gamma_{x,v}(r)$ and $\tilde{t} = t + r(\eta - t)$.

$$I_1 \leq \int_0^{\frac{1}{2}} \int_B \int_0^{\eta} \int_{\Psi(B)} \Big(M r^{-1/2} \left|\nabla_v g\right| (\tilde{t}, \tilde{x}, \tilde{v}) \varphi(y, w) - \left|\nabla_v g\right|^2 (\tilde{t}, \tilde{x}, \tilde{v}) \varphi^2(y, w) \Big)_+ \frac{1}{1-r} \left|\det \mathcal{B}(r)\right|^{-1} \mathrm{d}(\tilde{x}, \tilde{v}) \mathrm{d}\tilde{t} \mathrm{d}(y, w) \mathrm{d}r$$

I_1

Substitute $(\tilde{x}, \tilde{v}) = \Psi_{r,t,\eta,y,w}(x, v) := \gamma_{x,v}(r)$ and $\tilde{t} = t + r(\eta - t)$.

$$I_1 \leq \int_0^{\frac{1}{2}} \int_B \int_0^{\eta} \int_{\tilde{B}} \Big(M r^{-1/2} \left|\nabla_v g\right| (\tilde{t}, \tilde{x}, \tilde{v}) \varphi(y, w) - \left|\nabla_v g\right|^2 (\tilde{t}, \tilde{x}, \tilde{v}) \varphi^2(y, w) \Big)_+ \mathrm{d}(\tilde{x}, \tilde{v}) \mathrm{d}\tilde{t} \mathrm{d}(y, w) \mathrm{d}r$$

as $\Psi(B) \subset \tilde{B}$ and $\det \mathcal{B}(r) \sim 1$ on $(\frac{1}{2}, 1)$.

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Substitute $(\tilde{x}, \tilde{v}) = \Psi_{r,t,\eta,y,w}(x, v) := \gamma_{x,v}(r)$ and $\tilde{t} = t + r(\eta - t)$.

$$I_1 \leq \int_0^{\frac{1}{2}} \int_B \int_0^{\eta} \int_{\tilde{B}} \Big(M r^{-1/2} \left|\nabla_v g\right| (\tilde{t}, \tilde{x}, \tilde{v}) \varphi(y, w) - \left|\nabla_v g\right|^2 (\tilde{t}, \tilde{x}, \tilde{v}) \varphi^2(y, w) \Big)_+ \mathrm{d}(\tilde{x}, \tilde{v}) \mathrm{d}\tilde{t} \mathrm{d}(y, w) \mathrm{d}r$$

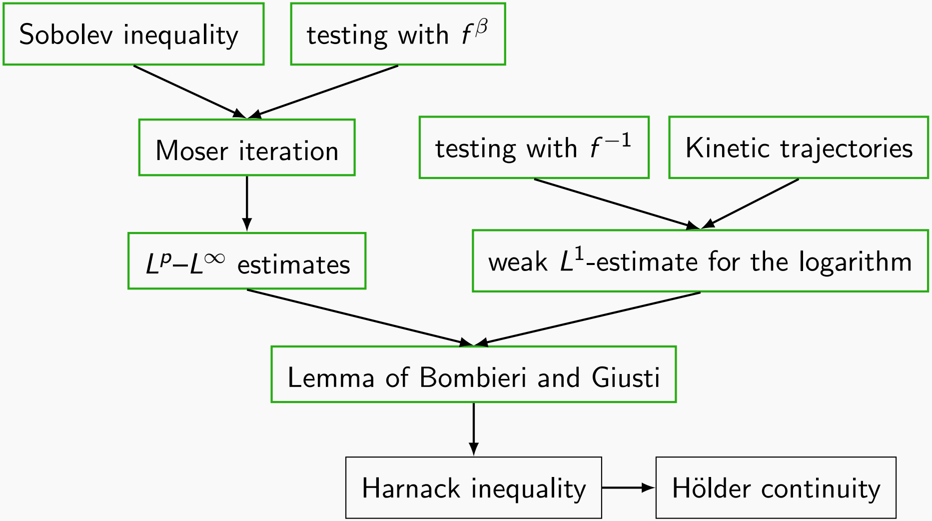
as $\Psi(B) \subset \tilde{B}$ and $\det \mathcal{B}(r) \sim 1$ on $(\frac{1}{2}, 1)$.

Calculating the r-integral from 0 to $\min\{1/2, M^2/p^2\}$ yields

$$\int_0^{1/2} \left(r^{-1/2} M p - p^2 \right)_+ \mathrm{d}r \lesssim M^2$$

for all $p > 0$. Here $p = \left|\nabla_v g\right| (\tilde{t}, \tilde{x}, \tilde{v}) \varphi(y, w)$.

Moser's 1971 method in kinetic theory

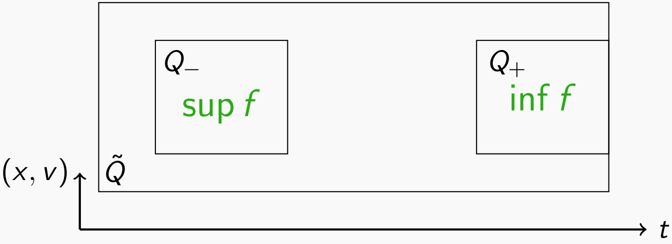


Harnack inequality

Theorem (DMNZ 24):

There exists a universal const $C = C(n, \lambda, \Lambda) > 0$ such that for any nonnegative weak solution f of (1) in $\tilde{Q}$ we have

$$\sup_{Q_-} f \leq C \inf_{Q_+} f.$$



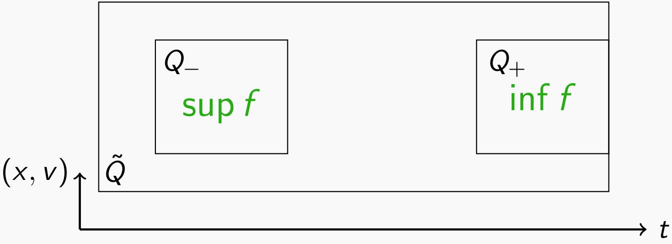
Harnack inequality

Theorem (DMNZ 24):

There exists a universal const $C = C(n) > 0$ such that for any nonnegative weak solution f of (1) in $\tilde{Q}$ we have

$$\sup_{Q_-} f \le C^{\mu} \inf_{Q_+} f.$$

Here, $\mu = \frac{1}{\lambda} + \Lambda$ if $\mathfrak{a}$ is symmetric. Optimal!



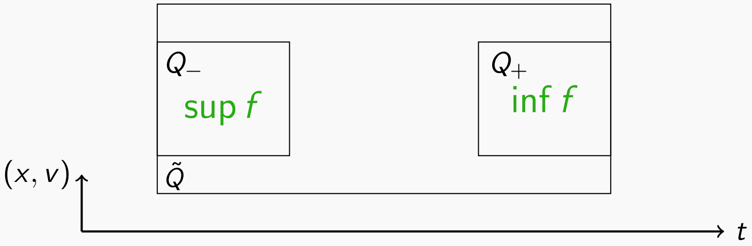
Weak Harnack inequality

Theorem (DMNZ 24):

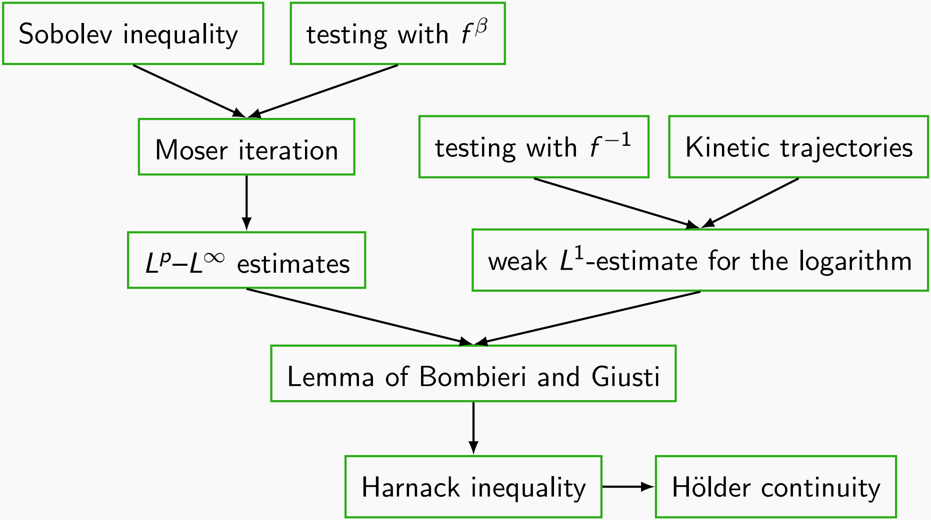
There exists a universal $C(n, \mu) > 0$ such that for all $p \in (0, 1 + \frac{1}{2n})$ and any nonnegative weak supersolution f to (1) in $\tilde{Q}$ we have

$$\left(\int_{Q_-} |f|^p \, \mathrm{d}(t, x, v) \right)^p \leq C \inf_{Q_+} f.$$

Optimal range for p.



Moser's 1971 method in kinetic theory



Euclidean smoothing

$$f = f(v) \mapsto \int_{\mathbb{R}^n} f(\mathsf{m})\varphi^2\left(\frac{v-\mathsf{m}}{r}\right) r^{-n}\mathrm{d}\mathsf{m} = \int_{\mathbb{R}^n} f(v - r\mathsf{m})\varphi^2(\mathsf{m})\mathrm{d}\mathsf{m}x$$

Parabolic smoothing

Space

$$f = f(t, v) \mapsto \int_{\mathbb{R}^n} f(t - sr, v - r^{1/2}\mathrm{m})\varphi^2(\mathrm{m})\mathrm{d}\mathrm{m}$$

Spacetime

$$f = f(t, v) \mapsto \int_{\mathbb{R}^{1+n}} f(t - sr, v - r^{1/2}\mathrm{m})\psi^2(s, \mathrm{m})\mathrm{d}(s, \mathrm{m})$$

Kinetic smoothing

Consider $\gamma^{(s,\mathrm{m})}\colon \mathbb{R} \to \mathbb{R}^{1+2n}$ with $\mathrm{m} = (\mathrm{m}_0, \mathrm{m}_1) \in \mathbb{R}^{2n}$, $s \neq 0$ defined as

$$\gamma^{(s,\mathrm{m})}(r;(t,x,v)) = \begin{pmatrix} \gamma_t^{(s,\mathrm{m})}(r) \\ \gamma_x^{(s,\mathrm{m})}(r) \\ \gamma_v^{(s,\mathrm{m})}(r) \end{pmatrix} = \begin{pmatrix} t + s\, r \\ \mathcal{A}_s(r) \begin{pmatrix} \mathrm{m}_0 \\ \mathrm{m}_1 \end{pmatrix} + \begin{pmatrix} 1 & s\, r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \end{pmatrix}$$

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Consider $\gamma^{(s,\mathrm{m})} \colon \mathbb{R} \to \mathbb{R}^{1+2n}$ with $\mathrm{m} = (\mathrm{m}_0, \mathrm{m}_1) \in \mathbb{R}^{2n}$, $s \neq 0$ defined as

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Space

$$[S_r(f)](t,x,v) = \frac{1}{c_\varphi} \int_B f(\gamma^{(s,\mathrm{m})}(r;(t,x,v))\varphi^2(\mathrm{m})\mathrm{d}\mathrm{m}$$

Spacetime

$$[T_r(f)](t,x,v) = \frac{1}{c_\psi} \int_Q f(\gamma^{(s,\mathrm{m})}(r;(t,x,v))\psi^2(s,\mathrm{m})\mathrm{d}(s,\mathrm{m})$$

Kinetic Sobolev embedding

Theorem (DMNZ 24):

Let $f \in L^2(\mathbb{R}^{1+n}; \dot{H}^1(\mathbb{R}^n))$ such that $\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot h$ for some $h \in L^2(\mathbb{R}^{1+2n}; \mathbb{R}^n)$, then

$$\|f\|_{L^{2\kappa}(\mathbb{R}^{1+2n})} \leq C \left(\|\nabla_v f\|_{L^2(\mathbb{R}^{1+2n})} + \|h\|_{L^2(\mathbb{R}^{1+2n})} \right)$$

with $\kappa = 1 + \frac{1}{2n}$ and $C = C(n) > 0$.

Kinetic Sobolev embedding

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with $\kappa = 1 + \frac{1}{2n}$ and $C = C(n) > 0$.

Local versions. No fundamental solution, only Young-type inequality.

Kinetic Nash inequality

Theorem (DMNZ 24):

Let $f \in L^2(\mathbb{R}^{1+n}; \dot{H}^1(\mathbb{R}^n)) \cap L^1(\mathbb{R}^{1+2n})$ such that we have $\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot h$ for some $h \in L^2(\mathbb{R}^{1+2n}; \mathbb{R}^n)$, then

$$\|f\|_{L^2(\mathbb{R}^{1+2n})}^{1+\frac{2}{2+4d}} \leq C\sqrt{\|\nabla_v f\|_{L^2(\mathbb{R}^{1+2n})}^2 + \|h\|_{L^2(\mathbb{R}^{1+2n})}^2}\,\|f\|_{L^1(\mathbb{R}^{1+2n})}^{\frac{2}{2+4d}}$$

for some $C = C(n) > 0$.

Kinetic Nash inequality

Theorem (DMNZ 24):

Let $f \in L^2(\mathbb{R}^{1+n}; \dot{H}^1(\mathbb{R}^n)) \cap L^1(\mathbb{R}^{1+2n})$ such that we have $\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot h$ for some $h \in L^2(\mathbb{R}^{1+2n}; \mathbb{R}^n)$, then

$$\|f\|_{L^2(\mathbb{R}^{1+2n})}^{1+\frac{2}{2+4d}} \leq C\sqrt{\|\nabla_v f\|_{L^2(\mathbb{R}^{1+2n})}^2 + \|h\|_{L^2(\mathbb{R}^{1+2n})}^2}\,\|f\|_{L^1(\mathbb{R}^{1+2n})}^{\frac{2}{2+4d}}$$

for some $C = C(n) > 0$.

Consequence of Sobolev and interpolation.

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