Mollification in kinetic geometry

Lukas Niebel
University of Münster

joint work with
Helge Dietert, Université Paris Cité
Clément Mouhot, University of Cambridge
Rico Zacher, Ulm University

Calderón–Zygmund Analysis Seminar, University of Chicago – May 4th, 2026

Elliptic Sobolev inequality

Theorem:

For $1 < p < d$ and $p^* = \frac{dp}{d-p}$ we have for f suitable

$$\|f\|_{\mathrm{L}^{p^*}(\mathbb{R}^d)} \leq C\|\nabla f\|_{\mathrm{L}^p(\mathbb{R}^d)}.$$

Elliptic geometry

Laplace equation on $\mathbb{R}^d$:

$$-\Delta f = 0$$

Translation invariance for $h \in \mathbb{R}^d$:

$$x \circ h = x + h \text{ and } \Delta(f(\cdot + h))(x) = [\Delta f](x + h)$$

Scaling invariance for $\lambda > 0$:

$$\delta_\lambda x = \lambda x \text{ and } \Delta(f(\lambda\cdot))(x) = \lambda^2[\Delta f](\lambda x)$$

Trajectories $\gamma = \gamma(\tau)\colon [0, \infty) \to \mathbb{R}^d$ are straight lines.

For a direction $\mathrm{m} = m_1 \in \mathbb{R}^d$ and a starting point $x \in \mathbb{R}^d$:

$$\gamma^{\mathrm{m}}(\tau) = \gamma^{\mathrm{m}}(\tau; x) = x - \tau\mathrm{m}, \qquad \gamma^{\mathrm{m}}(0) = x, \qquad \dot{\gamma}^{\mathrm{m}}(\tau) = -\mathrm{m}$$

Elliptic mollification

Choose $\psi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\psi \geq 0, \quad \int_{\mathbb{R}^d} \psi(\mathrm{m})\,\mathrm{d}\mathrm{m} = 1, \quad \operatorname{supp}\psi \subset \{\mathrm{m} \in \mathbb{R}^d : 1 < |\mathrm{m}| < 2\}.$$

For $\tau > 0$ define the elliptic mollification kernel

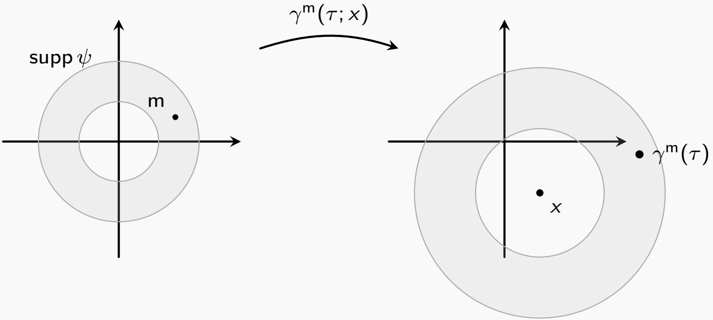
$$K_\tau(y) := \tau^{-d}\psi\Big(\frac{y}{\tau}\Big).$$

Mollification is a trajectory average at scale τ:

$$\begin{aligned}
[T_{K_\tau} f](x) &:= \int_{\mathbb{R}^d} f(x-y) K_\tau(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} f(x - \tau\mathrm{m})\,\psi(\mathrm{m})\,\mathrm{d}\mathrm{m} \\
&= \int_{\mathbb{R}^d} f(\gamma^{\mathrm{m}}(\tau; x))\,\psi(\mathrm{m})\,\mathrm{d}\mathrm{m}
\end{aligned}$$

Elliptic mollification as averaging along trajectories

$$[T_{K_\tau} f](x) = \int_{\mathbb{R}^d} f(\gamma^{\mathsf{m}}(\tau; x))\, \psi(\mathsf{m})\, \mathrm{d}\mathsf{m}$$



Representation formula

Apply the fundamental theorem of calculus along $r \mapsto \gamma^{\mathrm{m}}(r;x)$

$$
\begin{aligned}
f(x) - [T_{K_\tau} f](x) &= \int_{\mathbb{R}^d} \big(f(x) - f(x - \tau\mathrm{m})\big)\psi(\mathrm{m})\,\mathrm{d}\mathrm{m} \\
&= \int_{\mathbb{R}^d} \int_0^\tau \mathrm{m} \cdot \nabla f(x - r\mathrm{m})\,\mathrm{d}r\,\psi(\mathrm{m})\,\mathrm{d}\mathrm{m} \\
&= \int_{\mathbb{R}^d} \nabla f(x-y) \cdot \int_0^\tau r^{-d}\psi\Big(\frac{y}{r}\Big)\,\frac{y}{r}\,\mathrm{d}r\,\mathrm{d}y.
\end{aligned}
$$

Kernel

$$
\mathcal{B}_\tau(y) := \int_0^\tau r^{-d}\psi\Big(\frac{y}{r}\Big)\,\frac{y}{r}\,\mathrm{d}r
$$

$$
[T_{\mathcal{B}_\tau} F](x) := \int_{\mathbb{R}^d} F(x-y) \cdot \mathcal{B}_\tau(y)\,\mathrm{d}y
$$

Representation

$$
f = f - T_{K_\tau} f + T_{K_\tau} f = T_{\mathcal{B}_\tau}(\nabla f) + T_{K_\tau} f
$$

Young inequalities

For a kernel J set

$$[T_J g](x) = \int_{\mathbb{R}^d} J(x-y)\cdot g(y)\,\mathrm{d}y.$$

Young inequality ($1 \leq p, q, \theta \leq \infty$):

$$\|T_J g\|_{\mathsf{L}^q} \leq \|J\|_{\mathsf{L}^\theta}\,\|g\|_{\mathsf{L}^p}, \qquad \frac{1}{q} + 1 = \frac{1}{\theta} + \frac{1}{p}$$

Weak Young inequality ($1 < p, q, \theta < \infty$):

$$\|T_J g\|_{\mathsf{L}^q} \lesssim \|J\|_{\mathsf{L}^{\theta,\infty}}\,\|g\|_{\mathsf{L}^p}, \qquad \frac{1}{q} + 1 = \frac{1}{\theta} + \frac{1}{p}$$

Kernel estimates

$$\mathcal{B}_\tau(y) = \int_0^\tau r^{-d}\psi\Big(\frac{y}{r}\Big)\ \frac{y}{r}\,\mathrm{d}r$$

Critical scale $\theta = \frac{d}{d-1}$

(i) Scaling of the mollifier

$$\|K_\tau\|_{\mathrm{L}^\theta(\mathbb{R}^d)} = \tau^{d(\frac{1}{\theta}-1)}\|\psi\|_{\mathrm{L}^\theta(\mathbb{R}^d)} \le C\tau^{-1}$$

(change of variables)

(ii) Critical bound for the integrated kernel

$$|\mathcal{B}_\tau(y)| \le C\int_{|y|/2}^{|y|} r^{-d}\,\mathrm{d}r \le C\,|y|^{1-d}\,, \qquad \mathcal{B}_\tau(y) = 0 \text{ for } |y| \ge 2\tau$$

$$\|\mathcal{B}_\tau\|_{\mathrm{L}^{\frac{d}{d-1},\infty}(\mathbb{R}^d)} \le C$$

($\operatorname{supp}\psi(y/r)$; $\|f\|_{\mathrm{L}^{p,\infty}} := \sup_{\lambda>0}\lambda\big|\{x\in X : |f(x)|>\lambda\}\big|^{\frac{1}{p}}$)

Elliptic Sobolev inequality

Theorem:

For $1 < p < d$ and $p^* = \frac{dp}{d-p}$ we have for f suitable

$$\|f\|_{\mathsf{L}^{p^*}(\mathbb{R}^d)} \leq C\|\nabla f\|_{\mathsf{L}^p(\mathbb{R}^d)}.$$

Proof:

Note $\frac{1}{p^*} + 1 = \frac{1}{p} + \frac{1}{\frac{d}{d-1}}$, and write

$$f = (f - T_{K_\tau} f) + T_{K_\tau} f.$$

Use the (weak) Young inequality:

$$\|f - T_{K_\tau} f\|_{\mathsf{L}^{p^*}} = \|T_{\mathcal{B}_\tau}(\nabla f)\|_{\mathsf{L}^{p^*}} \leq C\|\mathcal{B}_\tau\|_{\mathsf{L}^{\frac{d}{d-1},\infty}} \|\nabla f\|_{\mathsf{L}^p} \leq C\|\nabla f\|_{\mathsf{L}^p}$$

$$\|T_{K_\tau} f\|_{\mathsf{L}^{p^*}} = \|K_\tau * f\|_{\mathsf{L}^{p^*}} \leq \|K_\tau\|_{\mathsf{L}^{\frac{d}{d-1}}} \|f\|_{\mathsf{L}^p} \leq C\tau^{-1}\|f\|_{\mathsf{L}^p}$$

$$\|f\|_{\mathsf{L}^{p^*}} \leq C\|\nabla f\|_{\mathsf{L}^p} + C\tau^{-1}\|f\|_{\mathsf{L}^p}, \qquad \tau \to \infty$$

Parabolic geometry

Parabolic equation on $\mathbb{R}^{1+d}$:

$$\partial_t f = \Delta_x f$$

Translation invariance for $h = (h_0, h_1) \in \mathbb{R}^{1+d}$:

$$(t, x) \circ h = (t + h_0, x + h_1)$$

Parabolic scaling for $\lambda > 0$:

$$\delta_\lambda(t, x) = (\lambda^2 t, \lambda x), \qquad \mathrm{Q} = 2 + d$$

Parabolic trajectories for $\mathrm{m} = (m_0, m_1) \in \mathbb{R}^{1+d}$, $\tau \in [0, \infty)$:

$$\gamma^{\mathrm{m}}(\tau; (t, x)) = (t - \tau^2\, m_0,\ x - \tau\, m_1)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\gamma^{\mathrm{m}}(\tau; (t, x)) = (-2\tau m_0,\ -m_1)$$

Parabolic regularity

Parabolic equation on $\mathbb{R}^{1+d}$:

$$\partial_t f = \nabla_x \cdot (\nabla_x f)$$

Weak (energy) solutions:

$$f \in \mathsf{L}^\infty_t \, \mathsf{L}^2_x \cap \mathsf{L}^2_t \, \dot{\mathsf{H}}^1_x$$

Parabolic regularity:

$$S \quad \text{in} \quad \partial_t f = \nabla_x \cdot S \qquad \text{and} \qquad \nabla_x f$$

Parabolic mollification

$\gamma^{\mathrm{m}}(\tau;(t,x)) = (t-\tau^2 m_0, x-\tau m_1)$

Choose $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^{1+d})$ such that

$$\psi \geq 0, \qquad \int_{\mathbb{R}^{1+d}} \psi(\mathrm{m})\,\mathrm{d}\mathrm{m} = 1, \qquad \operatorname{supp}\psi \subset (1,2)\times B_1(0).$$

Averaging along parabolic trajectories at scale τ:

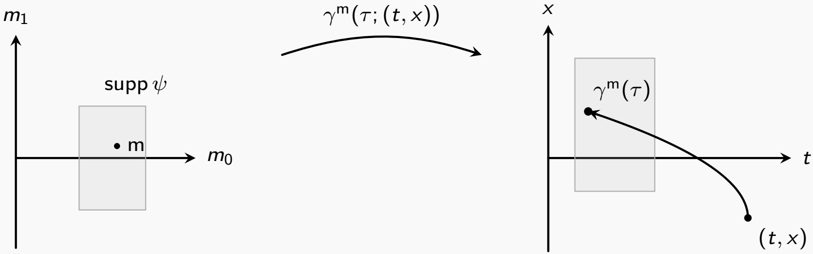
$$\begin{aligned}[T_{K_\tau} f](t,x) &:= \int_{\mathbb{R}^{1+d}} f(\gamma^{\mathrm{m}}(\tau;(t,x)))\,\psi(\mathrm{m})\,\mathrm{d}\mathrm{m} \\ &= \int_{\mathbb{R}^{1+d}} f(t-s,x-y) K_\tau(s,y)\,\mathrm{d}s\,\mathrm{d}y\end{aligned}$$

where for $\tau > 0$ we define the parabolic mollifier

$$K_\tau(s,y) := \tau^{-(d+2)}\psi\Big(\frac{s}{\tau^2},\frac{y}{\tau}\Big).$$

Mollification as averaging along parabolic trajectories

$$[T_{K_\tau} f](t,x) = \int_{\mathbb{R}^{1+d}} f\big(\gamma^{\mathsf{m}}(\tau;(t,x))\big)\,\psi(\mathsf{m})\,\mathrm{d}\mathsf{m}$$



Representation formula I

$\gamma^{\mathsf{m}}(r;(t,x)) = (t - r^2 m_0, x - r m_1)$

Apply the fundamental theorem of calculus to $r \mapsto f(\gamma^{\mathsf{m}}(r;(t,x)))$:

$$
\begin{aligned}
& f(t,x) - T_{K_\tau} f(t,x) \\
&= -\int_{\mathbb{R}^{1+d}} \int_0^\tau \frac{\mathrm{d}}{\mathrm{d}r} f(\gamma^{\mathsf{m}}(r;(t,x)))\,\mathrm{d}r\,\psi(\mathsf{m})\,\mathrm{d}\mathsf{m} \\
&= \int_0^\tau \int_{\mathbb{R}^{1+d}} \Big(2 r m_0\,\partial_t f + m_1 \cdot \nabla_x f\Big)(\gamma^{\mathsf{m}}(r;(t,x)))\,\psi(\mathsf{m})\,\mathrm{d}\mathsf{m}\,\mathrm{d}r \\
&=: I_S(t,x) + I_{\nabla f}(t,x)
\end{aligned}
$$

Representation formula II

For I_S, insert $\partial_t f = \nabla_x \cdot S$ and change variables:

$$s = r^2 m_0, \qquad y = r m_1, \qquad \mathrm{dm} = r^{-Q}\,\mathrm{d}s\,\mathrm{d}y.$$

$$\begin{aligned}
I_S(t,x) &= \int_0^\tau \int_{\mathbb{R}^{1+d}} 2 r m_0\, \partial_t f(\gamma^{\mathrm{m}}(r;(t,x)))\,\psi(\mathrm{m})\,\mathrm{dm}\,\mathrm{d}r \\
&= \int_0^\tau \int_{\mathbb{R}^{1+d}} 2 r m_0\,(\nabla_x \cdot S)(t - r^2 m_0, x - r m_1)\,\psi(\mathrm{m})\,\mathrm{dm}\,\mathrm{d}r \\
&= \int_0^\tau \int_{\mathbb{R}^{1+d}} 2\frac{s}{r}\, r^{-Q} (\nabla_x \cdot S)(t-s, x-y)\,\psi\Big(\frac{s}{r^2}, \frac{y}{r}\Big)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}r \\
&= -\int_0^\tau \int_{\mathbb{R}^{1+d}} 2\frac{s}{r}\, r^{-Q} (\nabla_y \cdot S)(t-s, x-y)\,\psi\Big(\frac{s}{r^2}, \frac{y}{r}\Big)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}r \\
&= \int_{\mathbb{R}^{1+d}} S(t-s, x-y) \cdot 2\int_0^\tau r^{-Q} \frac{s}{r^2} \nabla_{m_1} \psi\Big(\frac{s}{r^2}, \frac{y}{r}\Big)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}y \\
&= \int_{\mathbb{R}^{1+d}} S(t-s, x-y) \cdot \mathcal{P}_\tau(s,y)\,\mathrm{d}s\,\mathrm{d}y
\end{aligned}$$

Representation formula III

Forcing term:

$$I_{\nabla f}(t,x) = \int_0^\tau \int_{\mathbb{R}^{1+d}} m_1 \cdot \nabla_x f(t - r^2 m_0, x - r m_1)\, \psi(\mathrm{m})\, \mathrm{d}\mathrm{m}\, \mathrm{d}r$$
$$= \int_{\mathbb{R}^{1+d}} \nabla_x f(t-s, x-y) \cdot \left[\int_0^\tau r^{-\mathsf{Q}} \frac{y}{r} \psi\Big(\frac{s}{r^2}, \frac{y}{r}\Big)\, \mathrm{d}r \right] \mathrm{d}s\, \mathrm{d}y$$

Kernels

$$\mathcal{P}_\tau(s,y) := 2 \int_0^\tau r^{-\mathsf{Q}} \frac{s}{r^2} \nabla_{m_1} \psi\Big(\frac{s}{r^2}, \frac{y}{r}\Big)\, \mathrm{d}r$$

$$\mathcal{B}_\tau(s,y) := \int_0^\tau r^{-\mathsf{Q}} \frac{y}{r} \psi\Big(\frac{s}{r^2}, \frac{y}{r}\Big)\, \mathrm{d}r$$

Representation

$$f = f - T_{K_\tau} f + T_{K_\tau} f = T_{\mathcal{P}_\tau} S + T_{\mathcal{B}_\tau}(\nabla_x f) + T_{K_\tau} f$$

Kernel estimates

Critical scale $\theta = \frac{d+2}{d+1} = \frac{Q}{Q-1}$

(i) Pointwise bound

$$|\mathcal{P}_\tau(s,y)| + |\mathcal{B}_\tau(s,y)| \le C \frac{\mathbb{1}_{\{0<s<2\tau^2\}} \mathbb{1}_{\{|y|\le\sqrt{s}\}}}{s^{\frac{d+1}{2}}}$$

(ii) Weak bound

$$\|\mathcal{P}_\tau\|_{\mathsf{L}^{\theta,\infty}(\mathbb{R}^{1+d})} + \|\mathcal{B}_\tau\|_{\mathsf{L}^{\theta,\infty}(\mathbb{R}^{1+d})} \le C$$

(iii) Mollifier scaling

$$\|K_\tau\|_{\mathsf{L}^{\theta}(\mathbb{R}^{1+d})} = C\tau^{-1}$$

(on $\operatorname{supp} \psi(s/r^2, y/r)$ one has $r^2 < s < 2r^2$ and $|y| < r$, hence $r \sim \sqrt{s}$;
(iii) is the parabolic change of variables $(s,y) = (\tau^2\sigma, \tau\eta)$)

Parabolic Sobolev inequality

Theorem:

For $p \in (1, \mathrm{Q})$ set $p^* = \frac{\mathrm{Q}p}{\mathrm{Q}-p}$. All suitable f, S with $\partial_t f = \nabla_x \cdot S$ satisfy

$$\|f\|_{\mathrm{L}^{p^*}(\mathbb{R}^{1+d})} \leq C\big(\|\nabla_x f\|_{\mathrm{L}^p(\mathbb{R}^{1+d})} + \|S\|_{\mathrm{L}^p(\mathbb{R}^{1+d})}\big).$$

Proof:

$$f = (f - T_{K_\tau} f) + T_{K_\tau} f, \qquad 1 + \frac{1}{p^*} = \frac{1}{p} + \frac{1}{\theta}, \qquad \theta = \frac{d+2}{d+1}.$$

$$\begin{aligned}
\|f - T_{K_\tau} f\|_{\mathrm{L}^{p^*}} &\leq \|T_{\mathcal{P}_\tau} S\|_{\mathrm{L}^{p^*}} + \|T_{\mathcal{B}_\tau}(\nabla_x f)\|_{\mathrm{L}^{p^*}} \\
&\leq C\big(\|\mathcal{P}_\tau\|_{\mathrm{L}^{\theta,\infty}} \|S\|_{\mathrm{L}^p} + \|\mathcal{B}_\tau\|_{\mathrm{L}^{\theta,\infty}} \|\nabla_x f\|_{\mathrm{L}^p}\big) \\
&\leq C\big(\|S\|_{\mathrm{L}^p} + \|\nabla_x f\|_{\mathrm{L}^p}\big),
\end{aligned}$$

$$\|T_{K_\tau} f\|_{\mathrm{L}^{p^*}} \leq \|K_\tau\|_{\mathrm{L}^\theta} \|f\|_{\mathrm{L}^p} \leq C\tau^{-1} \|f\|_{\mathrm{L}^p}$$

$$\|f\|_{\mathrm{L}^{p^*}} \leq C\big(\|S\|_{\mathrm{L}^p} + \|\nabla_x f\|_{\mathrm{L}^p}\big) + C\tau^{-1}\|f\|_{\mathrm{L}^p}, \qquad \tau \to \infty$$

Kinetic Landau equation

For $t \geq 0$, $x \in \Omega \subset \mathbb{R}^d$, $v \in \mathbb{R}^d$, the density $f = f(t,x,v)$ solves

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{Q}(f)$$

where

$$\mathcal{Q}(f) = \nabla_v \cdot \left(\int_{\mathbb{R}^d} A(v - v_*) \Big(f(v_*) \nabla_v f(v) - f(v) \nabla_{v_*} f(v_*) \Big) \, \mathrm{d}v_* \right)$$

and for $z \in \mathbb{R}^d$

$$A(z) = a_{d,\gamma} |z|^\gamma \big(|z|^2 \, \mathrm{Id}_d - z \otimes z \big), \quad a_{d,\gamma} > 0, \quad -d \leq \gamma \leq 1.$$

(we dropped the (t,x) dependence of f)

Kinetic regularity

Kolmogorov equation on $\mathbb{R}^{1+2d}$:

$$(\partial_t + v \cdot \nabla_x)f = \nabla_v \cdot \big(\mathfrak{a}(t,x,v)\nabla_v f\big)$$

Weak (energy) solutions

$$f \in \mathsf{L}^\infty_t \, \mathsf{L}^2_{x,v} \cap \mathsf{L}^2_{t,x} \, \mathsf{H}^1_v$$

Kinetic regularity

$$S \quad \text{in} \quad (\partial_t + v \cdot \nabla_x)f = \nabla_v \cdot S \qquad \text{and} \qquad \nabla_v f$$

Kinetic geometry

Kolmogorov equation on $\mathbb{R}^{1+2d}$:

$$(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot (\nabla_v f)$$

Translation invariance (kinetic group structure):

$$(t, x, v) \circ (s, y, w) = (t + s,\ x + y + s\, v,\ v + w)$$

Scaling invariance for $\lambda > 0$:

$$\delta_\lambda(t, x, v) = (\lambda^2 t,\ \lambda^3 x,\ \lambda v), \qquad \mathrm{Q} = 4d + 2$$

Kinetic trajectories will be considered later.

Hörmander's commutator observation:

$$[\partial_{v_i},\ \partial_t + v \cdot \nabla_x] = \partial_{x_i}, \qquad i = 1, \ldots, d$$

Kinetic mollification (naive)

$\mathsf{Q} = 4d + 2$

$$[T_{K_\tau} f](t,x,v) = \int_{\mathbb{R}^{1+2d}} K_\tau\big((t,x,v)^{-1} \circ \mathrm{m}\big)\, f(\mathrm{m})\, \mathrm{dm}$$

with

$$K_\tau^{\mathrm{na}}(s,y,w) = \tau^{-\mathsf{Q}} \psi\Big(\frac{s}{\tau^2}, \frac{y}{\tau^3}, \frac{w}{\tau}\Big)$$

where $\psi \in \mathrm{C}_c^\infty$, $\psi \geq 0$, $\int \psi = 1$, $\operatorname{supp} \psi \subset (-2,-1) \times B_1(0) \times B_1(0)$

$$(t,x,v)^{-1} \circ \mathrm{m} = \big(m_0 - t,\ m_1 - x - (m_0 - t)v,\ m_2 - v\big)$$

Why the naive mollifier is not good enough

After the change of variables, the naive trajectory average is along

$$\gamma^{\mathrm{m}}(r;(t,x,v)) = (t,x,v)\circ(r^2 m_0, r^3 m_1, r m_2),$$
$$T_{K_\tau^{\mathrm{na}}} f(t,x,v) = \int f(\gamma^{\mathrm{m}}(\tau;(t,x,v)))\psi(\mathrm{m})\,\mathrm{dm}.$$

These curves are not tangent to the kinetic directions $\partial_t + v\cdot\nabla_x$ and ∇_v.

If $(\partial_t + v\cdot\nabla_x)f = \nabla_v\cdot S$, the representation formula is

$$f(t,x,v) - T_{K_\tau^{\mathrm{na}}} f(t,x,v) = I_{\mathrm{kin}}(t,x,v) + I_x(t,x,v),$$

$$I_{\mathrm{kin}}(t,x,v) = -\int_0^\tau\int\Big(2rm_0\,\nabla_v\cdot S + m_2\cdot\nabla_v f\Big)(\gamma^{\mathrm{m}}(r;(t,x,v)))\psi(\mathrm{m})\,\mathrm{dm}\,\mathrm{d}r,$$

$$I_x(t,x,v) = -\int_0^\tau\int r^2\big(3m_1 - 2m_0 m_2\big)\cdot\nabla_x f(\gamma^{\mathrm{m}}(r;(t,x,v)))\psi(\mathrm{m})\,\mathrm{dm}\,\mathrm{d}r.$$

Kinetic mollification

$\mathrm{Q} = 4d + 2$

$$[T_{K_\tau} f](t,x,v) = \int_{\mathbb{R}^{1+2d}} K_\tau\big((t,x,v)^{-1} \circ \mathrm{m}\big)\, f(\mathrm{m})\, \mathrm{d}\mathrm{m}$$

with

$$K_\tau(s,y,w) = 2^d \tau^{-\mathrm{Q}} \psi\left(\frac{s}{\tau^2}, \mathcal{A}_{\frac{s}{\tau^2}}(\tau)^{-1}\begin{pmatrix} y \\ w \end{pmatrix}\right)$$

where $\psi \in \mathrm{C}_c^\infty$, $\psi \geq 0$, $\int \psi = 1$, $\operatorname{supp} \psi \subset (-2,-1) \times B_1(0) \times B_1(0)$

and

$$\mathcal{A}_{m_0}(\tau)^{-1} = \begin{pmatrix} \dfrac{\sin(\log\tau) - 3\cos(\log\tau)}{\tau^3}\ \mathrm{Id}_d & \dfrac{2m_0\cos(\log\tau)}{\tau}\ \mathrm{Id}_d \\ \dfrac{3\sin(\log\tau) + \cos(\log\tau)}{m_0\tau^3}\ \mathrm{Id}_d & -\dfrac{2\sin(\log\tau)}{\tau}\ \mathrm{Id}_d \end{pmatrix}$$

$$(t,x,v)^{-1} \circ \mathrm{m} = \big(m_0 - t,\ m_1 - x - (m_0 - t)v,\ m_2 - v\big)$$

Kinetic trajectories

Consider $\gamma = (\gamma_t, \gamma_x, \gamma_v) \colon [0,1] \to \mathbb{R}^{1+2d}$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} f(\gamma(\tau)) = \dot{\gamma}_t(\tau)\big[\partial_t f\big](\gamma(\tau)) + \dot{\gamma}_x(\tau) \cdot [\nabla_x f](\gamma(\tau)) + \dot{\gamma}_v(\tau) \cdot [\nabla_v f](\gamma(\tau))$$

Kinetic trajectories

Consider $\gamma = (\gamma_t, \gamma_x, \gamma_v)\colon [0,1] \to \mathbb{R}^{1+2d}$ with $\dot\gamma_x = \dot\gamma_t \gamma_v$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau} f(\gamma(\tau)) &= \dot\gamma_t(\tau)\big[\partial_t f\big](\gamma(\tau)) + \dot\gamma_x(\tau)\cdot[\nabla_x f](\gamma(\tau)) + \dot\gamma_v(\tau)\cdot[\nabla_v f](\gamma(\tau)) \\
&= \dot\gamma_t(\tau)\big[(\partial_t + v\cdot\nabla_x) f\big](\gamma(\tau)) + \dot\gamma_v(\tau)\cdot[\nabla_v f](\gamma(\tau))
\end{aligned}
$$

We call γ a kinetic trajectory if

$$\dot\gamma_x(\tau) = \dot\gamma_t(\tau)\gamma_v(\tau).$$

Kinetic trajectories via Newton laws

Construct $\gamma = (\gamma_t, \gamma_x, \gamma_v)\colon [0,\infty) \to \mathbb{R}^{1+2d}$ with $\gamma(0) = (t, x, v)$ and $\dot\gamma_x(\tau) = \dot\gamma_t(\tau)\gamma_v(\tau)$.

Ansatz:

$$\gamma_t^{\mathrm{m}}(\tau) = t + m_0\tau^2$$

$$\dot\gamma_t^{\mathrm{m}}(\tau) = 2m_0\tau$$

$$\dot\gamma_v^{\mathrm{m}}(\tau) = \frac{1}{m_0}\partial_\tau\left(\frac{\dot g_1(\tau)}{2\tau}\right)m_1 + \partial_\tau\left(\frac{\dot g_2(\tau)}{2\tau}\right)m_2$$

for $\mathrm{m} = (m_0, m_1, m_2) \in \mathbb{R}^{1+2d}$ and $g_1, g_2\colon [0,\infty) \to \mathbb{R}$

Solving Newton's laws gives

$$\gamma_v^{\mathrm{m}}(\tau) = v + \frac{\dot g_1(\tau)}{2m_0\tau}m_1 + \frac{\dot g_2(\tau)}{2\tau}m_2$$

$$\gamma_x^{\mathrm{m}}(\tau) = x + m_0\tau^2 v + g_1(\tau)m_1 + m_0 g_2(\tau)m_2$$

Kinetic trajectories via Newton laws

Let $\mathsf{m} = (m_0, m_1, m_2) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^d \times \mathbb{R}^d$
and $g_1, g_2 \colon [0, \infty) \to \mathbb{R}$ then

$$\gamma^{\mathsf{m}}(\tau) = \begin{pmatrix} t + m_0\tau^2 \\ \underbrace{\begin{pmatrix} g_1(\tau)\,\mathrm{Id}_d & m_0 g_2(\tau)\,\mathrm{Id}_d \\ \frac{\dot{g}_1(\tau)}{2m_0\tau}\,\mathrm{Id}_d & \frac{\dot{g}_2(\tau)}{2\tau}\,\mathrm{Id}_d \end{pmatrix}}_{\mathcal{A}_{m_0}(\tau)} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} \mathrm{Id}_d & m_0\tau^2\,\mathrm{Id}_d \\ 0_d & \mathrm{Id}_d \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \end{pmatrix}$$

is such that $\gamma^{\mathsf{m}}(0) = (t, x, v)$ and $\dot{\gamma}^{\mathsf{m}}_x(\tau) = \dot{\gamma}^{\mathsf{m}}_t(\tau)\gamma^{\mathsf{m}}_v(\tau)$.

Choice of the forcings

Let $\mathrm{m} = (m_0, m_1, m_2) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^d \times \mathbb{R}^d$ and choose

$$g_1(\tau) = \tau^3 \sin(\log \tau), \qquad g_2(\tau) = \tau^3 \cos(\log \tau) \qquad (\tau > 0).$$

$$\gamma^{\mathrm{m}}(\tau) = \begin{pmatrix} t + m_0\tau^2 \\ \begin{pmatrix} \tau^3 \sin(\log \tau)\ \mathrm{Id}_d & m_0\tau^3 \cos(\log \tau)\ \mathrm{Id}_d \\ \frac{\tau}{2m_0}\big(3\sin(\log \tau) + \cos(\log \tau)\big)\ \mathrm{Id}_d & \frac{\tau}{2}\big(3\cos(\log \tau) - \sin(\log \tau)\big)\ \mathrm{Id}_d \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \dots \end{pmatrix}$$

is such that $\gamma^{\mathrm{m}}(0) = (t, x, v)$ and $\dot\gamma_x^{\mathrm{m}}(\tau) = \dot\gamma_t^{\mathrm{m}}(\tau)\gamma_v^{\mathrm{m}}(\tau)$.

Kinetic trajectories are controllable

Let (t_0, x_0, v_0) and (t_1, x_1, v_1) with $t_1 \neq t_0$.

Set

$$m_0 = t_1 - t_0.$$

Since the endpoint matrix is invertible,

$$\det \mathcal{A}_{m_0}(1) = (-2)^{-d} \neq 0,$$

we choose the parameters by solving the endpoint equation:

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \mathcal{A}_{m_0}(1)^{-1} \left[\begin{pmatrix} x_1 \\ v_1 \end{pmatrix} - \begin{pmatrix} \mathrm{Id}_d & m_0 \, \mathrm{Id}_d \\ 0_d & \mathrm{Id}_d \end{pmatrix} \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \right]$$

$$\gamma^{\mathrm{m}}(1; (t_0, x_0, v_0)) = (t_1, x_1, v_1)$$

Mollification as averaging along trajectories

Rewrite

$$\gamma^{\mathsf{m}}(\tau;(t,x,v)) = (t,x,v) \circ \left(m_0\tau^2,\, \mathcal{A}_{m_0}(\tau)\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \right)$$

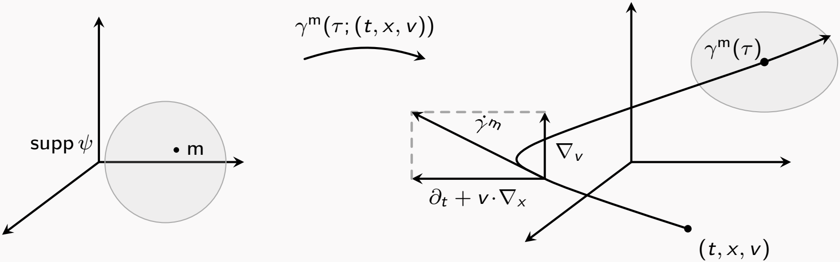
Recall

$$K_\tau(s,y,w) = 2^d \tau^{-\mathsf{Q}} \psi\left(\frac{s}{\tau^2}, \mathcal{A}_{\frac{s}{\tau^2}}(\tau)^{-1}\begin{pmatrix} y \\ w \end{pmatrix}\right), \qquad \mathsf{Q} = 4d+2$$

$$\begin{aligned}
[T_{K_\tau} f](t,x,v) &= \int_{\mathbb{R}^{1+2d}} K_\tau\big((t,x,v)^{-1}\circ(s,y,w)\big)\, f(s,y,w)\; \mathrm{d}(s,y,w) \\
&= \int_{\mathbb{R}^{1+2d}} f(\gamma^{\mathsf{m}}(\tau;(t,x,v)))\, \psi(\mathsf{m})\, \mathrm{d}\mathsf{m}
\end{aligned}$$

Mollification as averaging along trajectories

$$[T_{K_\tau} f](t,x,v) = \int_{\mathbb{R}^{1+2d}} f(\gamma^{\mathsf{m}}(\tau;(t,x,v)))\,\psi(\mathsf{m})\,\mathrm{d}\mathsf{m}.$$



Kinetic mollification

$\mathrm{Q} = 4d + 2$

$$[T_{K_\tau} f](t,x,v) = \int_{\mathbb{R}^{1+2d}} K_\tau\big((t,x,v)^{-1} \circ \mathrm{m}\big)\, f(\mathrm{m})\, \mathrm{dm}$$

with

$$K_\tau(s,y,w) = 2^d \tau^{-\mathrm{Q}} \psi\left(\frac{s}{\tau^2}, \mathcal{A}_{\frac{s}{\tau^2}}(\tau)^{-1}\begin{pmatrix} y \\ w \end{pmatrix}\right)$$

where $\psi \in \mathrm{C}_c^\infty$, $\psi \geq 0$, $\int \psi = 1$, $\operatorname{supp}\psi \subset (-2,-1) \times B_1(0) \times B_1(0)$

and

$$\mathcal{A}_{m_0}(\tau)^{-1} = \begin{pmatrix} \dfrac{\sin(\log\tau) - 3\cos(\log\tau)}{\tau^3}\ \mathrm{Id}_d & \dfrac{2m_0\cos(\log\tau)}{\tau}\ \mathrm{Id}_d \\ \dfrac{3\sin(\log\tau) + \cos(\log\tau)}{m_0\tau^3}\ \mathrm{Id}_d & -\dfrac{2\sin(\log\tau)}{\tau}\ \mathrm{Id}_d \end{pmatrix}$$

$$(t,x,v)^{-1} \circ \mathrm{m} = \big(m_0 - t,\ m_1 - x - (m_0 - t)v,\ m_2 - v\big)$$

Kinetic mollification

$$[T_{K_\tau} f](t,x,v) = \int_{\mathbb{R}^{1+2d}} K_\tau\big((t,x,v)^{-1} \circ \mathrm{m}\big)\, f(\mathrm{m})\, \mathrm{dm}$$

with

$$K_\tau(s,y,w) = 2^d \tau^{-\mathsf{Q}} \psi\left(\frac{s}{\tau^2}, \mathcal{A}_{\frac{s}{\tau^2}}(\tau)^{-1} \begin{pmatrix} y \\ w \end{pmatrix}\right)$$

where $\psi \in \mathrm{C}_c^\infty$, $\psi \geq 0$, $\int \psi = 1$, $\operatorname{supp} \psi \subset (-2,-1) \times B_1(0) \times B_1(0)$

and $\mathcal{A}_{m_0}(\tau)\colon [0,\infty) \to \mathbb{R}^{2d \times 2d}$, $m_0 \neq 0$ with

- **(C1)** $\det \mathcal{A}_{m_0}(\tau) \sim \tau^{4d}$
- **(C2)** $\left|(\mathcal{A}_{m_0}(\tau)^{-1})_{\cdot;2}\right| \lesssim_{m_0} \tau^{-1}$
- **(C3)** $|\partial_\tau (\mathcal{A}_{m_0}(\tau))_{2;\cdot}| \lesssim_{m_0} 1$

Representation formula I

Assume $(\partial_t + v\cdot\nabla_x)f = \nabla_v\cdot S$. Along kinetic trajectories:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}f(\gamma^{\mathrm{m}}(r)) &= 2rm_0\,[(\partial_t + v\cdot\nabla_x)f](\gamma^{\mathrm{m}}(r)) + \dot{\gamma}_v^{\mathrm{m}}(r)\cdot[\nabla_v f](\gamma^{\mathrm{m}}(r))\\ &= 2rm_0\,[\nabla_v\cdot S](\gamma^{\mathrm{m}}(r)) + \dot{\gamma}_v^{\mathrm{m}}(r)\cdot[\nabla_v f](\gamma^{\mathrm{m}}(r))\end{aligned}$$

Apply the fundamental theorem of calculus to $r\mapsto f(\gamma^{\mathrm{m}}(r))$:

$$\begin{aligned}&f(t,x,v) - T_{K_\tau}f(t,x,v)\\ &= -\int_{\mathbb{R}^{1+2d}}\int_0^\tau \frac{\mathrm{d}}{\mathrm{d}r}f(\gamma^{\mathrm{m}}(r))\,\mathrm{d}r\,\psi(\mathrm{m})\,\mathrm{d}\mathrm{m}\\ &= -\int_0^\tau\int_{\mathbb{R}^{1+2d}}\Big(2rm_0\,[\nabla_v\cdot S] + \dot{\gamma}_v^{\mathrm{m}}(r)\cdot\nabla_v f\Big)(\gamma^{\mathrm{m}}(r))\psi(\mathrm{m})\,\mathrm{d}\mathrm{m}\,\mathrm{d}r\\ &=: I_S(t,x,v) + I_\nabla f(t,x,v)\end{aligned}$$

Representation formula II

Change of variables

$$(s,y,w) = \left(m_0 r^2,\, \mathcal{A}_{m_0}(r)\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}\right), \qquad \gamma^{\mathsf{m}}(r) = (t,x,v)\circ(s,y,w)$$

$$\mathsf{m} := \left(\frac{s}{r^2}, \mathcal{A}_{\frac{s}{r^2}}(r)^{-1}\begin{pmatrix} y \\ w \end{pmatrix}\right), \qquad \mathrm{d}\mathsf{m} = 2^d r^{-\mathsf{Q}}\,\mathrm{d}(s,y,w)$$

$$\begin{aligned}
&I_S(t,x,v)\\
&= -\int_0^\tau \int_{\mathbb{R}^{1+2d}} 2^{d+1} s\, r^{-\mathsf{Q}-1} [\nabla_v \cdot S]((t,x,v)\circ(s,y,w))\psi(\mathsf{m}(s,y,w))\,\mathrm{d}(s,y,w)\,\mathrm{d}r\\
&= -\int_0^\tau \int_{\mathbb{R}^{1+2d}} 2^{d+1} s\, r^{-\mathsf{Q}-1} \nabla_w \cdot \big[S((t,x,v)\circ(s,y,w))\big]\psi(\mathsf{m}(s,y,w))\,\mathrm{d}(s,y,w)\,\mathrm{d}r\\
&= \int_{\mathbb{R}^{1+2d}} S((t,x,v)\circ(s,y,w)) \cdot \mathcal{P}_\tau(s,y,w)\,\mathrm{d}(s,y,w).
\end{aligned}$$

$$\mathcal{P}_\tau(s,y,w) = 2^{d+1}\int_0^\tau r^{-\mathsf{Q}}\frac{s}{r^2}[\nabla_{(m_1,m_2)}\psi]^T\left(\frac{s}{r^2}, \mathcal{A}_{\frac{s}{r^2}}(r)^{-1}\begin{pmatrix} y \\ w \end{pmatrix}\right)\left(\mathcal{A}_{\frac{s}{r^2}}(r)^{-1}\right)_{\cdot;2} r\,\mathrm{d}r$$

Representation formula III

Same change of variables

$$I_{\nabla f}(t,x,v) = \int_{\mathbb{R}^{1+2d}} [\nabla_v f]((t,x,v)\circ(s,y,w)) \cdot \mathcal{B}_\tau(s,y,w)\,\mathrm{d}(s,y,w),$$

$$\mathcal{B}_\tau(s,y,w) := -2^d \int_0^\tau r^{-\mathsf{Q}} \psi\left(\frac{s}{r^2}, \mathcal{A}_{\frac{s}{r^2}}(r)^{-1}\begin{pmatrix} y \\ w \end{pmatrix}\right) \times \left[\partial_\rho\big(\mathcal{A}_{\frac{s}{r^2}}(\rho)\big)_{2;\cdot}\Big|_{\rho=r} \mathcal{A}_{\frac{s}{r^2}}(r)^{-1}\begin{pmatrix} y \\ w \end{pmatrix}\right] \mathrm{d}r.$$

Kernel estimates

Critical scale $\theta = \frac{4d+2}{4d+1} = \frac{Q}{Q-1}$, $\qquad Q = 4d + 2$

(i) Pointwise bound

$$|\mathcal{P}_\tau(s,y,w)| + |\mathcal{B}_\tau(s,y,w)| \lesssim \frac{\mathbb{1}_{\{-2\tau^2<s<0\}}\mathbb{1}_{\{|y|\lesssim|s|^{3/2}\}}\mathbb{1}_{\{|w|\lesssim|s|^{1/2}\}}}{|s|^{\frac{Q-1}{2}}}$$

(on the support of the integrand one has $\sqrt{|s|/2} \le r \le \sqrt{|s|}$, $|y| \lesssim r^3$, $|w| \lesssim r$; since the r-integrand is $\lesssim r^{-Q}$, integrating in r gives $|s|^{-(Q-1)/2}$)

(ii) Weak bound

$$\|\mathcal{P}_\tau\|_{\mathsf{L}^{\theta,\infty}} + \|\mathcal{B}_\tau\|_{\mathsf{L}^{\theta,\infty}} \le C$$

(iii) Mollifier scaling

$$\|K_\tau\|_{\mathsf{L}^\theta} = C\tau^{-1}$$

Young inequalities for kinetic convolution

Recall

$$[T_J g](t,x,v) = \int_{\mathbb{R}^{1+2d}} J((t,x,v)^{-1} \circ \mathrm{m})\, g(\mathrm{m})\, \mathrm{dm}$$
$$= \int_{\mathbb{R}^{1+2d}} J(s,y,w)\, g((t,x,v) \circ (s,y,w))\, \mathrm{d}(s,y,w)$$

The kinetic shifts are measure preserving.

Young inequality ($1 \leq p, q, \theta \leq \infty$):

$$\|T_J g\|_{\mathsf{L}^q} \leq \|J\|_{\mathsf{L}^\theta} \|g\|_{\mathsf{L}^p}, \qquad \frac{1}{q} + 1 = \frac{1}{\theta} + \frac{1}{p}$$

Weak Young inequality ($1 < p, q, \theta < \infty$):

$$\|T_J g\|_{\mathsf{L}^q} \lesssim \|J\|_{\mathsf{L}^{\theta,\infty}} \|g\|_{\mathsf{L}^p}, \qquad \frac{1}{q} + 1 = \frac{1}{\theta} + \frac{1}{p}$$

Kinetic Sobolev inequality: proof

Let $1 < p < Q$, $p^* = \frac{pQ}{Q-p}$, and $\theta = \frac{Q}{Q-1}$.

Representation:

$$f = f - T_{K_\tau} f + T_{K_\tau} f = T_{\mathcal{P}_\tau}(S) + T_{\mathcal{B}_\tau}(\nabla_v f) + T_{K_\tau} f,$$

Kernel estimates:

$$\|\mathcal{P}_\tau\|_{\mathsf{L}^{\theta,\infty}} + \|\mathcal{B}_\tau\|_{\mathsf{L}^{\theta,\infty}} \lesssim 1.$$

Weak Young inequality:

$$\|f - T_{K_\tau} f\|_{\mathsf{L}^{p^*}} \lesssim \|S\|_{\mathsf{L}^p} + \|\nabla_v f\|_{\mathsf{L}^p}.$$

Mollified part as $\tau \to \infty$:

$$\|T_{K_\tau} f\|_{\mathsf{L}^{p^*}} \leq \|K_\tau\|_{\mathsf{L}^\theta} \|f\|_{\mathsf{L}^p} \lesssim \tau^{-1} \|f\|_{\mathsf{L}^p} \longrightarrow 0.$$

Kinetic Sobolev inequality

Theorem (Dietert–Mouhot–N.-Zacher '25)[1]

$$Q = 4d + 2, \qquad 1 < p < Q, \qquad p^* = \frac{pQ}{Q - p}.$$

Let $f \in \mathrm{L}^p(\mathbb{R}^{1+2d})$ with $\nabla_v f \in \mathrm{L}^p(\mathbb{R}^{1+2d})$ and suppose

$$(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot S,$$

with $S \in \mathrm{L}^p(\mathbb{R}^{1+2d}; \mathbb{R}^d)$. Then

$$\|f\|_{\mathrm{L}^{p^*}(\mathbb{R}^{1+2d})} \lesssim \|\nabla_v f\|_{\mathrm{L}^p(\mathbb{R}^{1+2d})} + \|S\|_{\mathrm{L}^p(\mathbb{R}^{1+2d})}.$$

[1] Earlier versions and related works: Hörmander '67, Pascucci–Polidoro '04, Golse–Imbert–Mouhot–Vasseur '19, Guerand–Mouhot '22, Auscher–Imbert-N. '24, Auscher–N. '26.

Transfer of regularity: Hörmander

Hörmander's Besov seminorm in the x-variable:

$$\|f\|_{\mathsf{L}^p_{t,v}\,\dot{\mathsf{B}}^{1/3}_{p,\infty,x}} := \sup_{h\neq 0} \frac{\|\Delta^h_x f\|_{\mathsf{L}^p_{t,x,v}}}{|h|^{1/3}}.$$

Theorem (N. '26)[1]

$$\mathsf{Q} = 4d+2, \qquad p \in \left[\frac{\mathsf{Q}}{\mathsf{Q}-1}, \infty\right], \qquad \frac{1}{q} = \frac{1}{p} + \frac{1}{\mathsf{Q}}.$$

Let $f \in \mathcal{S}(\mathbb{R}^{1+2d}) \cap \mathsf{L}^p$ satisfy

$$(\partial_t + v\cdot\nabla_x)f = \nabla_v \cdot S_0 + S_1,$$

with $\nabla_v f, S_0 \in \mathsf{L}^p$ and $S_1 \in \mathsf{L}^q$. Then

$$\|f\|_{\mathsf{L}^p_{t,v}\,\dot{\mathsf{B}}^{1/3}_{p,\infty,x}} \lesssim \|\nabla_v f\|_{\mathsf{L}^p} + \|S_0\|_{\mathsf{L}^p} + \|S_1\|_{\mathsf{L}^q}.$$

[1] Related works and earlier versions: Hörmander '67, Albritton–Armstrong–Mourrat-Novack '24.

Transfer of regularity: Bouchut

Theorem (Auscher-N. '26, N. '26)[1]

$$\mathrm{Q} = 4d + 2, \qquad p \in \Big(\frac{\mathrm{Q}}{\mathrm{Q}-1}, \infty\Big), \qquad \frac{1}{q} = \frac{1}{p} + \frac{1}{\mathrm{Q}}.$$

Let $f \in \mathcal{S}(\mathbb{R}^{1+2d}) \cap \mathrm{L}^p$ satisfy

$$(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot S_0 + S_1,$$

with $\nabla_v f, S_0 \in \mathrm{L}^p$ and $S_1 \in \mathrm{L}^q$. Then

$$\|D_x^{1/3} f\|_{\mathrm{L}^p_{t,x,v}} \lesssim \|\nabla_v f\|_{\mathrm{L}^p_{t,x,v}} + \|S_0\|_{\mathrm{L}^p_{t,x,v}} + \|S_1\|_{\mathrm{L}^q_{t,x,v}}.$$

[1] Related works and earlier versions: Bouchut '02, N.–Zacher '21, Auscher–Imbert-N. '24.

Kinetic Sobolev inequality: nonlocal diffusion I

Bessel-type control:

$$\widehat{D_v^s g}(\xi) = |\xi|^s \widehat{g}(\xi), \qquad 0 < s < 1.$$

$$\mathrm{Q} = 2s + 2d(s+1), \qquad 1 < p < \frac{\mathrm{Q}}{s}, \qquad \frac{1}{p_s^*} = \frac{1}{p} - \frac{s}{\mathrm{Q}}.$$

Theorem (Auscher-N. '26, N. '26)[1]

Assume

$$(\partial_t + v \cdot \nabla_x) f = D_v^s S, \qquad D_v^s f,\ S \in \mathrm{L}^p(\mathbb{R}^{1+2d}).$$

Then:

$$\|f\|_{\mathrm{L}^{p_s^*}} \lesssim \|D_v^s f\|_{\mathrm{L}^p} + \|S\|_{\mathrm{L}^p}.$$

[1] Related works and earlier versions: Imbert–Silvestre '20, Pascucci–Pesce '24, Auscher–Imbert–N. '24., Anceschi–Palatucci–Piccinini '24

Kinetic Sobolev inequality: nonlocal diffusion II

Gagliardo-type fractional control:

$$\mathfrak{D}_v^s f(t,x,v,h) = \frac{f(t,x,v+h) - f(t,x,v)}{|h|^s}, \qquad \mathrm{d}\eta(h) = \frac{\mathrm{d}h}{|h|^d}$$

$$\mathsf{Q} = 2s + 2d(s+1), \qquad 1 < p < \frac{\mathsf{Q}}{s}, \qquad \frac{1}{p_s^*} = \frac{1}{p} - \frac{s}{\mathsf{Q}}$$

Theorem (N. '26)

Assume

$$(\partial_t + v \cdot \nabla_x) f = \mathfrak{D}_v^{s,*} S, \qquad \mathfrak{D}_v^s f,\ S \in \mathsf{L}^p(\mathrm{d}\eta \otimes \mathrm{d}(t,x,v))$$

Then for every $q \in (p, p_s^*)$,

$$\|f\|_{\mathsf{L}^q} \lesssim \|f\|_{\mathsf{L}^p}^{1-\vartheta_{p,q}} \left(\|\mathfrak{D}_v^s f\|_{\mathsf{L}^p(\mathrm{d}\eta \otimes \mathrm{d}(t,x,v))} + \|S\|_{\mathsf{L}^p(\mathrm{d}\eta \otimes \mathrm{d}(t,x,v))} \right)^{\vartheta_{p,q}}$$

$$\vartheta_{p,q} = \frac{\frac{1}{p} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{p_s^*}}$$

Kinetic Harnack inequality

Kolmogorov equation with rough diffusion matrix

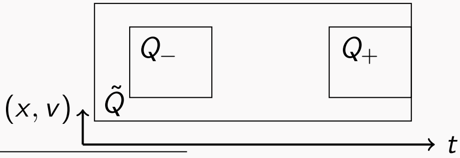
$$(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot (\mathfrak{a}(t,x,v) \nabla_v f)$$

$$0 < \lambda := \inf_{\substack{(t,x,v) \\ \xi \neq 0}} \frac{\langle \mathfrak{a}(t,x,v)\xi, \xi \rangle}{|\xi|^2} \qquad \Lambda := \sup_{\substack{(t,x,v) \\ \xi \neq 0}} \frac{|\mathfrak{a}(t,x,v)\xi|^2}{\langle \mathfrak{a}(t,x,v)\xi, \xi \rangle} < \infty$$

Theorem (Dietert–Mouhot–N.–Zacher '25)[1]

There exists a universal constant $C = C(d) > 0$ such that every nonnegative weak solution f in $\tilde{Q}$ satisfies

$$\sup_{Q_-} f \leq C^{\frac{1}{\lambda} + \Lambda} \inf_{Q_+} f.$$



[1] Related works and earlier versions: Wang–Zhang '09, Golse–Imbert–Mouhot–Vasseur '19, Guerand–Imbert '22, Guerand–Mouhot '22

Weak L^1-estimate for $\log f$

$(1)\ \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\mathfrak{a} \nabla_v f)$

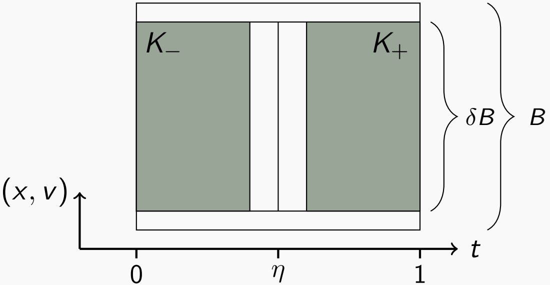
Theorem (Dietert–Mouhot–N.–Zacher '25):

Let $\delta, \eta \in (0,1)$. Then for any supersolution $f > 0$ to (1) there exists a constant $C = C(d, \delta, \eta, \lambda, \Lambda) > 0$ such that

$$s\,|\{(t,x,v) \in K_-\colon \log f(t,x,v) - c(f) > s\}| \leq C$$

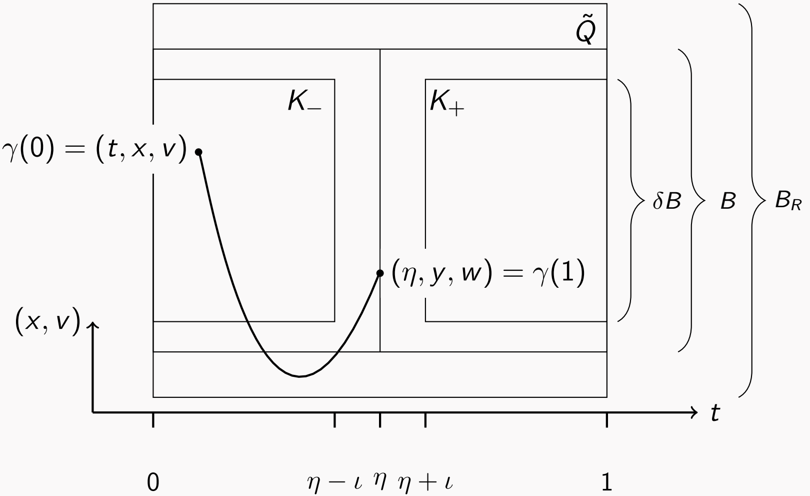
$$s\,|\{(t,x,v) \in K_+\colon c(f) - \log f(t,x,v) > s\}| \leq C$$

for all $s > 0$ with $c(f) = \dfrac{1}{c_\varphi} \displaystyle\int_B \log f(\eta, y, w) \varphi^2(y,w) \mathrm{d}(y,w)$.



Kinetic trajectories and the logarithm

(1) $\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\mathfrak{a} \nabla_v f)$



$$(\partial_t + v \cdot \nabla_x) \log f \geq \nabla_v \cdot (\mathfrak{a} \nabla_v \log f) + \langle \mathfrak{a} \nabla_v \log f, \nabla_v \log f \rangle.$$

Kinetic Gagliardo–Nirenberg inequality

Theorem (Dietert–N.–Zacher '26):

Let $p, \mu \in (1, \infty)$ and $q > \max\{2, p, \mu\}$ with

$$\frac{1}{q} = \frac{1}{4d+2}\left(\frac{3d+1}{p} + \frac{d+1}{\mu} - 1\right).$$

For $f \in \mathsf{L}^2(\mathbb{R}^{1+2d})$, $\nabla_v f \in \mathsf{L}^p(\mathbb{R}^{1+2d})$, and $S_0 \in \mathsf{L}^\mu(\mathbb{R}^{1+2d}; \mathbb{R}^d)$ with

$$(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot S_0$$

we have

$$\|f\|_{\mathsf{L}^q_{t,x,v}} \leq C \, \|\nabla_v f\|_{\mathsf{L}^p_{t,x,v}}^{\frac{3d+1}{4d+2}} \, \|S_0\|_{\mathsf{L}^\mu_{t,x,v}}^{\frac{d+1}{4d+2}}.$$

Proof via adapted kinetic trajectories.

Kinetic p-Laplace equation

$$(\partial_t + v \cdot \nabla_x) f - \nabla_v \cdot \left(|\nabla_v f|^{p-2} \nabla_v f \right) = 0$$

Theorem (Dietert–N.–Zacher '26):

Let $p \in \left(2 - \frac{2}{3d+2}, 2 + \frac{2}{d}\right)$ and let $f \geq 0$ be a weak subsolution in an open set containing the below cylinders. There exists $\varepsilon_0 = \varepsilon_0(d,p) > 0$ such that

$$\left.\begin{array}{l} \dfrac{1}{|Q_{K^{2-p},2R}|} \displaystyle\int_{Q_{K^{2-p},2R}} \left(\frac{f}{K}\right)^p \mathrm{d}(t,x,v) \leq \varepsilon_0, \quad p \geq 2, \\ \dfrac{1}{|Q_{K^{2-p},2R}|} \displaystyle\int_{Q_{K^{2-p},2R}} \left(\frac{f}{K}\right)^2 \mathrm{d}(t,x,v) \leq \varepsilon_0, \quad p < 2, \end{array}\right\} \Longrightarrow \operatorname*{ess\,sup}_{Q_{K^{2-p},R}} f \leq K.$$

Nonnegative weak subsolutions are locally bounded.

Intrinsic cylinders:

$$Q_{\theta,R} := \left\{ (t,x,v) : \ -\theta R^p \leq t < 0, \ x - tv \in B_{\theta R^{1+p}}(0), \ v \in B_R(0) \right\}$$

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lukasniebel.github.io