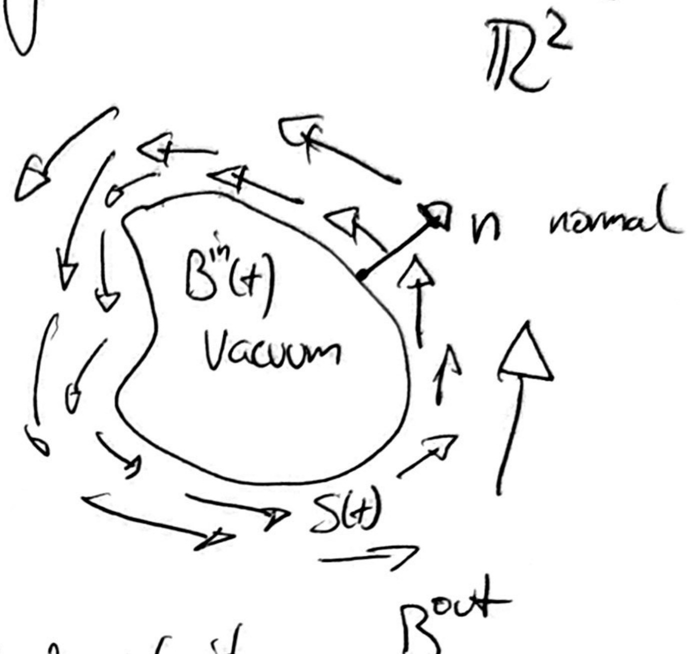
2D Bubbles with surface tension

$$\cancel{\partial_t} u + (u\cdot\nabla)u + \nabla P = 0 \quad B^{out}$$
$$\nabla\cdot u = 0 \quad B^{out}$$
Young–Laplace $\quad -P = \delta H \quad S$
$$u\cdot n = \cancel{V_n} 0 \; S$$



- $u(t,x) : [0,T]\times B^{out} \to \mathbb{R}^2$ fluid velocity
- H scalar signed curvature with $H_{\partial B_1} = 1$

($\gamma : [0,L] \to S$ arclength counter clockwise ($\tau = \gamma'$) $\quad \tau' = -Hn$ Frenet formula)
- $P(t,x)$ hydrodynamic pressure
- $\partial_t \gamma = V_n \cdot n + V_\tau \cdot \tau$

<u>Stationary</u> $\quad \partial_t u = 0 \quad ; \; V_n = 0 \; ;$ circulating flow $|u| \to 0$

[at infinity we don't see it]

Incompressible fluid:

Stream fct. $\quad$ scalar $\psi : B^{out} \to \mathbb{R}$

$$u = \nabla^\perp \psi = \begin{pmatrix} -\partial_2 \psi \\ \partial_1 \psi \end{pmatrix}$$

i) $\nabla\cdot u = -\partial_1\partial_2\psi + \partial_2\partial_1\psi = 0$

ii) $u\cdot\nabla\psi = \nabla^\perp\psi\cdot\nabla\psi = 0 \Rightarrow$ level sets of ψ are streamlines

$$0 = u\cdot n = \nabla^\perp\psi\cdot n = \nabla\psi\cdot\tau \Rightarrow \boxed{\psi = C_0 \text{ on } S}$$

(no physical meaning)

Vorticity $\omega := \partial_1 u_2 - \partial_2 u_1 := \operatorname{curl} u$

Irrotational $\omega = 0$

$$\omega = \underbrace{\partial_1 \partial_1 \psi - \partial_2(-\partial_2 \psi)}_{= \Delta\psi} = 0$$

$$\rightsquigarrow \boxed{-\Delta\psi = 0 \quad \text{in } B^{out}}$$

<u>Bernoulli's law</u>

steady Euler $(u\cdot\nabla)u + \nabla P = 0$

$$(u\cdot\nabla)u = \nabla\left(\tfrac{1}{2}|u|^2\right) + \underset{\operatorname{curl} u}{\overset{\shortparallel}{\omega}}\, u^\perp$$ $\Rightarrow$ u solves steady Euler

If $\omega = 0$ together

$$\nabla\left(P + \tfrac{1}{2}|u|^2\right) = 0 \Rightarrow P + \tfrac{1}{2}|u|^2 = B = \text{const}$$

Young-Laplace + $u = \nabla^\perp\psi$

$$\boxed{-\tfrac{1}{2}|\nabla\psi|^2 + \sigma H = \lambda \quad \text{on } S}$$

$\psi = c_0 \Rightarrow |\nabla\psi| = |\partial_n \psi|$

<u>Normalisation</u>

$$|B^{in}| = \pi R^2$$

3

Circulation

$$\Gamma_R = \int_{\partial B_R} u \cdot \tau \, ds \qquad R \gg 1, \quad \tau = e_\theta$$

counterclockwise unit tangent

If $-\Delta\psi = 0$ B^{out}
$\psi = c_0$ on S

$$\psi(x) = \alpha \log|x| + O(1), \quad |x| \to \infty \qquad (A)$$

$$\nabla\psi(x) = \alpha \frac{x}{|x|^2} + O(|x|^{-2})$$

$$u = \nabla^\perp \psi = \alpha \frac{e_\theta}{r} + O(r^{-2}) \quad \text{for } r = |x|$$

$$\Gamma_R = \int_0^{2\pi} \left(\alpha \cdot \frac{1}{R} + O(R^{-2})\right) R \, d\theta = 2\pi\alpha + O(R^{-1})$$

Rescale

$$x = R\tilde{x}, \quad \psi(x) = \alpha \tilde{\psi}(\tilde{x}) + \text{const}, \quad We = \frac{\alpha^2}{\sigma R} \text{ Weber}$$

Find $We \geq 0$, constants $c_0, \lambda \in \mathbb{R}$ a closed Jordan curve $S = \partial B^{in}$ with $|B^{in}| = \pi$ & $\psi : B^{out} \to \mathbb{R}$ s.t.

$We = 0$ CMC
$We = \infty$ Serrin-type

$P = P(B^{in}) = \text{Length}(S)$

$$\begin{aligned} -\Delta\psi &= 0 && B^{out} \\ \psi &= c_0 && S \\ \psi &= \log|x| + O(1) && |x| \to \infty \\ -\tfrac{1}{2} We |\nabla\psi|^2 + H &= \lambda && \text{on } S \end{aligned}$$

$|\nabla\psi| = \partial_n \psi$ by max principle

Unit circle

$B^{in} = B_1(0)$, $S = \partial B_1(0)$

$\psi_0(x) = C_0 + \lg|x|$

$\lambda = 1 - \frac{1}{2}We$

Non-circular solutions?

$\eta : [0,2\pi) \to (-1,\infty)$

$S_\eta = \{(1+\eta(\theta))\cancel{e_r(\theta)} : \theta \in [0,2\pi)\}$

$\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$

η in some Sobolev space with $\int_0^{2\pi}\eta(\theta)d\theta = 0$.

$G(\eta) = -\frac{1}{2}We\,|\nabla\psi_\eta|^2 + H_\eta$

$G(0) = -\frac{1}{2}We + 1$

$D_\eta G(\eta)\big|_{\eta=0}$ has a kernel if $We \in \{3,4,5,\ldots\}$

Question

Is there rigidity of the circular solution for large surface tension / small Radius / small circulation ?

$$We \ll 1$$

Theorem

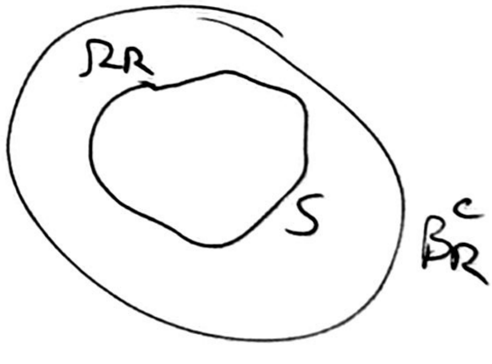
If S is a smooth solution and not a circle, then $We > 2$.

Lemma 1

$$\int_S (\partial_n \psi)\, ds = 2\pi$$

Proof

$$0 = \int_{\Omega_R} \Delta\psi\, dx \overset{\text{diverge thm}}{=} \underbrace{\int_{\partial B_R} \overset{= \frac{1}{r} + O(r^{-2})}{\partial_r \psi}\, ds}_{\longrightarrow 2\pi} - \int_S \partial_n \psi\, ds$$



$\square$

Lemma 2

$$\int_S H\, ds = 2\pi$$

(Gauss-Bonnet / turning number theorem)

Proof:

$\gamma : [0, L] \to \mathbb{R}^2$ arclength counterclockwise

$\tau = \gamma'$ unit tangent

$$= \begin{pmatrix} \cos \varphi(s) \\ \sin \varphi(s) \end{pmatrix} \qquad \varphi(L) - \varphi(0) = 2\pi$$

Frenet formula

$$-\varphi' n = \tau' = -H n$$

$$\int_S H\,ds = \int_0^L \varphi'(s)\,ds = 2\pi$$

$\square$

Lemma #2 (Pohozaev)

$$\int_S (x \cdot n)\,(\partial_n \psi)^2\,ds = 2\pi$$

Proof

Set $u = \psi - c_0$, $u = 0$ on S, $\Delta u = 0$ in B^{c}, $u = \log|x| + O(1)$, $|x| \to \infty$

Consider

$$X = (x \cdot \nabla u)\nabla u - \frac{1}{2}|\nabla u|^2 x$$

then

$$\nabla \cdot X = (x \cdot \nabla u)\Delta u = 0$$

Divergence Theorem

$$0 = \int_{\partial\Omega_R} X\cdot\gamma\, ds = \int_{\partial B_R} X\cdot e_r\, ds + \int_S X\cdot(-n)\, ds$$

On S: $u=0$ & $\nabla u = (\partial_n \psi)\cdot n$

$$\int_S X(-n)\, ds = \int_S (x\cdot(\partial_n\psi\, n))(\partial_n\psi\, n\cdot(-n)) - \frac{1}{2}(\partial_n\psi)^2 (x\cdot(-n))\, ds$$

$$= -\frac{1}{2}\int_S (x\cdot n)(\partial_n\psi)^2\, ds$$

$$\partial_r u = \frac{1}{r} + O(r^{-2}) \qquad X\cdot e_r = (x\cdot\nabla u)\partial_r u - \frac{1}{2}|\nabla u|^2 R$$

$$|\nabla u|^2 = \frac{1}{r^2} + O(r^{-3}) \qquad = \frac{1}{2R} + O(R^{-2})$$

$$x\cdot\nabla u = 1 + O(r^{-1})$$

$$\Rightarrow \int_{\partial B_R} X\cdot e_r\, ds \to \pi$$

$\square$

Lemma 4 (Minkowski identities) 8

1. $\int_S x \cdot n \, ds = 2\pi$ 2. $\int_S H(x \cdot n)\, ds = P\cancel{(B^{in})}$

1. $\int_S x \cdot n\, ds = \int_{B^{in}} \nabla \cdot x \, dx = 2\int_{B^{in}} dx = 2|B^{in}| = 2\pi$

2. $\gamma : [0,L] \to \mathbb{R}^2$ as before

$T = \gamma'$

$T' = -Hn$

$\frac{d}{ds}(\gamma \cdot T) = \gamma' \cdot T + \gamma \cdot T' = 1 - H(\gamma \cdot n) \quad \Big| \int_0^L \ldots ds$

$\square$

Proof of Theorem

$We = 0$ ✓ $We > 0$

1. Integrate jump

$$-\frac{1}{2} We \int (\partial_n \psi)^2 ds + 2\pi = \lambda P$$

2. Integrate jump $\times$ $(x \cdot n)$

$$-\frac{1}{2} We \underbrace{\int (\partial_n \psi)^2 x \cdot n\, ds}_{\underset{3}{=} 2\pi} + \underbrace{\int H (x \cdot n)\, ds}_{= P \; 4.2} = \underbrace{\lambda \int x \cdot n\, ds}_{\underset{4.1}{=} 2\pi}$$

$$\Rightarrow \lambda = \frac{P - \pi We}{2\pi}$$

$$\Rightarrow -\frac{1}{2} We \int_S (\partial_n \psi)^2 \, ds + 2\pi = P\left(\frac{P - \pi We}{2\pi}\right)$$

$$\Rightarrow \int_S (\partial_n \psi)^2 \, ds = P - \frac{P^2}{\pi We} + \frac{4\pi}{We}$$

$$(2\pi)^2 \underset{1}{=} \left(\int_S \partial_n \psi \, ds\right)^2 \le \left(\int_S 1^2 \, ds\right)\left(\int_S (\partial_n \psi)^2 \, ds\right) = P \int_S (\partial_n \psi)^2 \, ds$$

~~Ok~~

$$\Rightarrow 0 \le P - \frac{P^2}{\pi We} + \frac{4\pi}{We} - \frac{4\pi^2}{P}$$

$$= \frac{1}{\pi We P}\Big((P - 2\pi)(P + 2\pi)(\pi We - P)\Big)$$

Isoperimetric inqh: $P > 2\pi$

$$\Rightarrow \pi We - P \ge 0 \Rightarrow We \ge \frac{P}{\pi} > 2.$$

$\square$