Parabolic trajectories and the Harnack inequality

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Parabolic diffusion problem

Let $\Omega \subset \mathbb{R}^d$ open and $T > 0$. Consider weak solutions $u = u(t,x) \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$ to

$$\partial_t u = \nabla \cdot (A \nabla u) \quad \text{in } (0,T) \times \Omega$$

Parabolic diffusion problem with rough coefficients

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where $A = A(t, x)\colon (0, T) \times \Omega \to \mathbb{R}^{n \times n}$ is measurable, symmetric, bounded and

$$\lambda \left|\xi\right|^2 \le \langle A(t, x)\xi, \xi \rangle \le \Lambda \left|\xi\right|^2$$

for all $\xi \in \mathbb{R}^n$ a.e. $(t, x) \in (0, T) \times \Omega$. Set $\mu = \frac{1}{\lambda} + \Lambda$.

Parabolic geometry

Consider $A = \mathrm{Id}$.

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Translation: $(t_0, x_0) \mapsto (t - t_0, x - x_0)$
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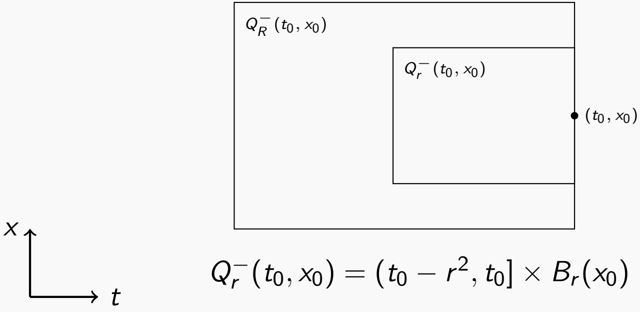
$L^2 - L^\infty$ estimate

$(1)\ \partial_t u = \nabla \cdot (A \nabla u)$

Theorem (Nash 58, Moser 64):

Let $\delta \in (0,1)$, $\delta \le r < R \le 1$, $t_0 \in (0,T)$, $x_0 \in \Omega$. There exists $c = c(d,\delta,\mu) > 0$ such that any pos. subsolution to (1) satisfies

$$\sup_{Q_r^-(t_0,x_0)} u^2 \le \frac{c}{(R-r)^{d+2}} \iint_{Q_R^-(t_0,x_0)} u^2 \mathrm{d}x\mathrm{d}t$$



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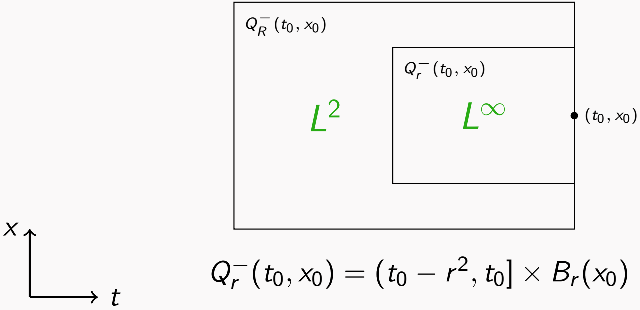

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The Harnack inequality

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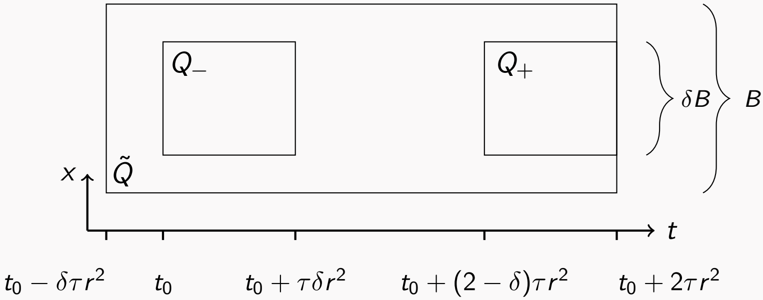
Let $\delta \in (0,1)$, $\tau > 0$. There exists $C = C(d, \delta, \tau) > 0$ such that for any nonnegative weak solution u of (1) in $\tilde{Q}$ we have

$$\sup_{Q_-} u \leq C^\mu \inf_{Q_+} u.$$



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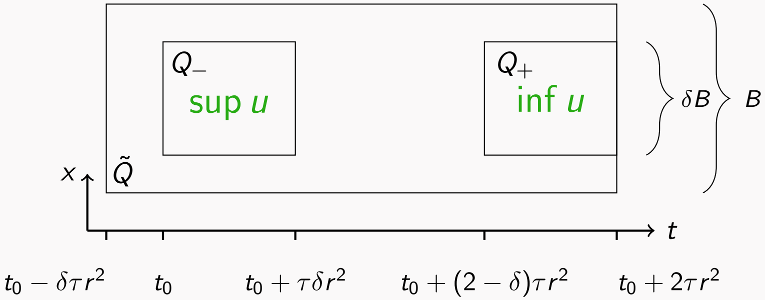

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- scaling and translation invariant
- implies Hölder continuity in (t, x) of u
- implies heat kernel bounds
- dependency of the constant on $\mu = \frac{1}{\lambda} + \Lambda$ is optimal

Brief history

- Harnack proves inequality for harmonic functions $\Delta u = 0$ in 1887
- Hadamard & Pini independently prove a Harnack inequality for the heat equation $\partial_t u = \Delta u$ in 1957
- De Giorgi solves Hilbert's 19th problem in 1957
  key step: a priori Hölder continuity for $-\nabla \cdot (A \nabla u) = 0$
- 1958 Nash treats the elliptic & parabolic eq. $\partial_t u = \nabla \cdot (A \nabla u)$

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- Moser proves Harnack inequality for the elliptic problem 1961
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- Bombieri and Giusti prove a Harnack inequality for elliptic differential equations on minimal surfaces in 1972
- ...and many more

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Proof of the Harnack inequality à la Moser 1971

3 Ingredients:

A: $L^p - L^\infty$ estimate for small $p \neq 0$

B: weak L^1-estimate for the logarithm of supersolutions

C: Lemma of Bombieri and Giusti

$L^p - L^\infty$ estimate

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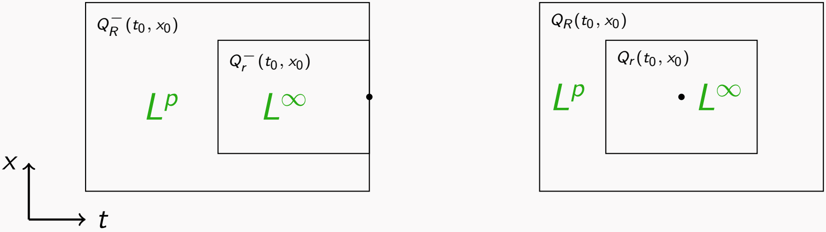

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Idea of the proof:

- test the equation (1) with $u^\beta \varphi^2$, $\beta < -1$
- employ the Sobolev inequality to obtain a gain of integrability on smaller cylinder
- iterate this inequality (Moser iteration)

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Weak L^1-estimate for $\log u$

Theorem (Moser 64 & 71):

Let $\delta, \eta \in (0,1)$ and $\varepsilon, \tau > 0$. Then for any supersolution $u \geq \varepsilon > 0$ to (1) there exists constants $c = c(u)$ and $C = C(d, \delta, \eta, \tau) > 0$ such t

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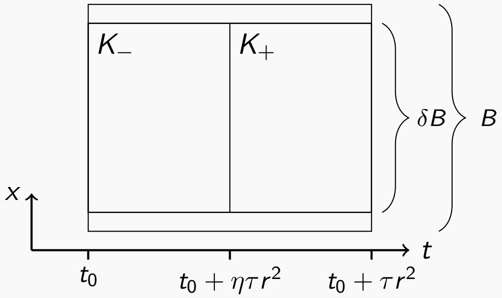

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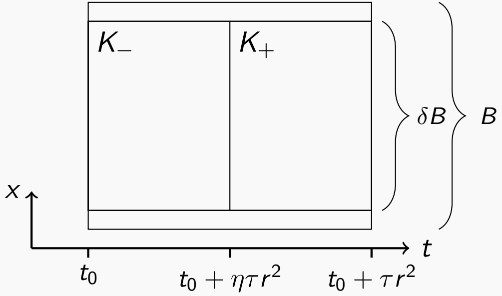

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- employ the spatial Poincaré inequality to obtain a differential inequality for

$$t \mapsto W(t) = \int_B \log u(t,y) \varphi^2(y) \mathrm{d}y$$

- several clever estimations yield the statement

Lemma of Bombieri and Giusti

Lemma (Moser 71, Bombieri and Giusti 72):

Let (X, ν) be a finite measure space, $U_\sigma \subset X$, $0 < \sigma \leq 1$ measurable with $U_{\sigma'} \subset U_\sigma$ if $\sigma' \leq \sigma$. Let $C_1, C_2 > 0$, $\delta \in (0, 1)$, $\tilde{\mu} > 1$, $\gamma > 0$. Suppose $0 \leq f \colon U_1 \to \mathbb{R}$ satisfies the following two conditions:

- for all $0 < \delta \leq r < R \leq 1$ and $0 < p < 1/\tilde{\mu}$ we have

$$\sup_{U_r} f^p \leq \frac{C_1}{(R - r)^\gamma \nu(U_1)} \int_{U_R} f^p \mathrm{d}\nu$$

- $s\nu(\{\log f > s\}) \leq C_2 \tilde{\mu}\ \nu(U_1)$ for all $s > 0$.

Then

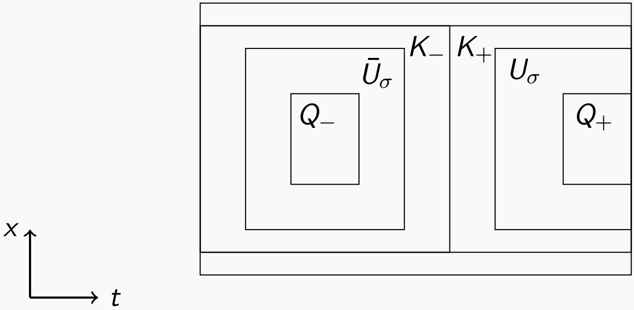
$$\sup_{U_\delta} f \leq C^{\tilde{\mu}}$$

where $C = C(C_1, C_2, \delta, \gamma)$.

Proof of the Harnack à la Moser 1971

Goal:

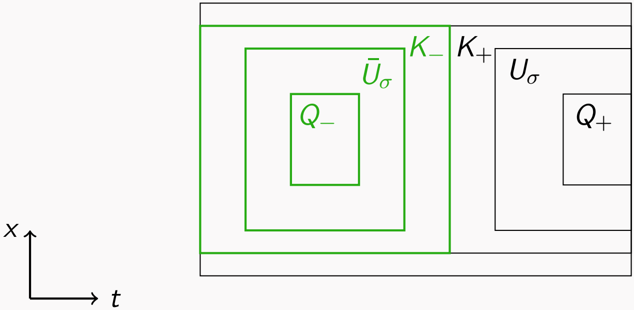
$$\sup_{Q_-} u \leq C^{\mu} \inf_{Q_+} u.$$



Proof of the Harnack inequality à la Moser 1971

Consider first $u \exp(-c(u))$ with $c(u)$ as in weak L^1-estimate.
Then the A,B and C combined give

$$\sup_{Q_-} u \leq e^{c(u)} \exp\left(C\mu\right)$$



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Consider now $u^{-1}\exp(c(u))$ with $c(u)$ as in weak L^1-estimate.
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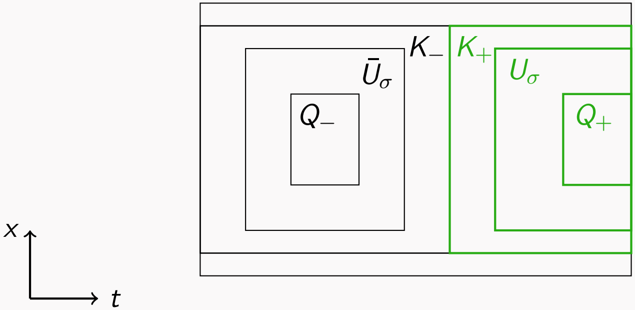

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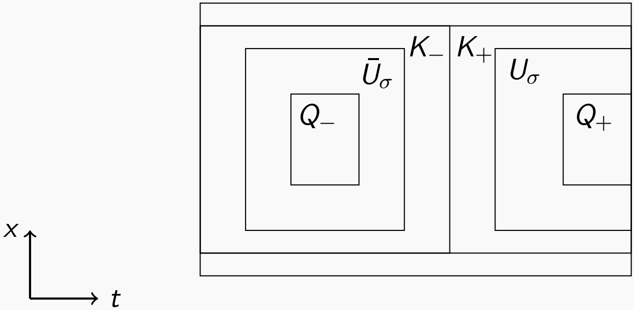
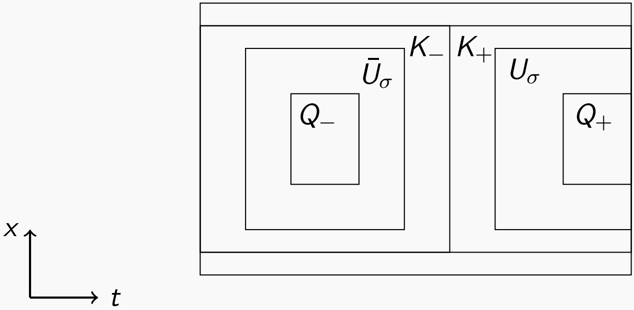


Proof of the Harnack inequality à la Moser 1971

$$\left.\begin{aligned} e^{c(u)} &\leq \exp\left(C\mu\right)\inf_{Q_+} u \\ \sup_{Q_-} u &\leq e^{c(u)}\exp\left(C\mu\right) \end{aligned}\right\} \Rightarrow \text{Harnack inequality}$$



Comments

- in comparision to De Giorgis, Nash's or Moser's old proof
  the method is much easier and less technical
- very robust
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  - discrete space problems (Delmotte 1999)
  - fractional (in time) equations (Zacher 2013)
  - non-local (in space) equations (Kassmann & Felsinger 2013)
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Problem:

The weak L^1-estimate heavily relies on a spatial Poincaré inequality.

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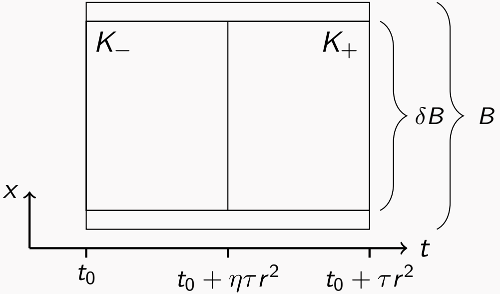

Weak L^1-estimate for $\log u$ modified

Theorem (N. & Zacher 22):

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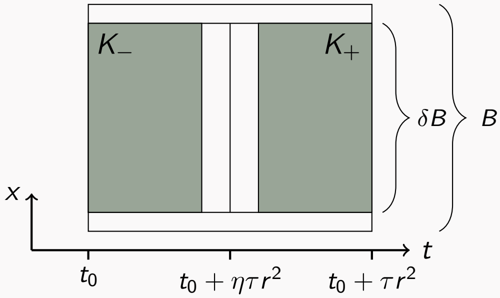
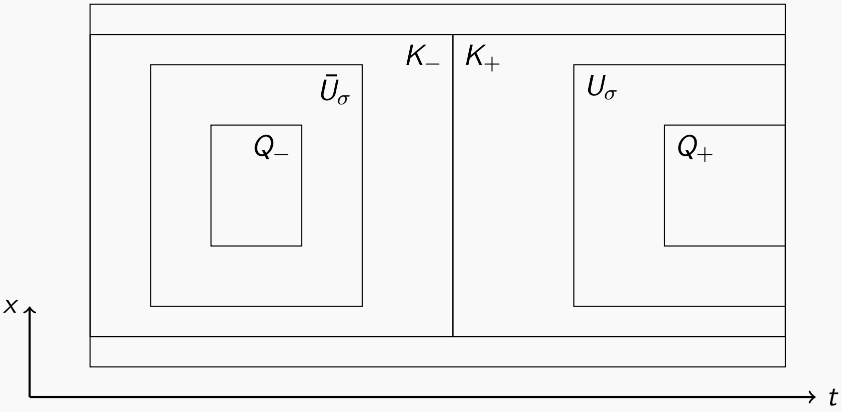


Proof of the Harnack inequality à la Moser 1971



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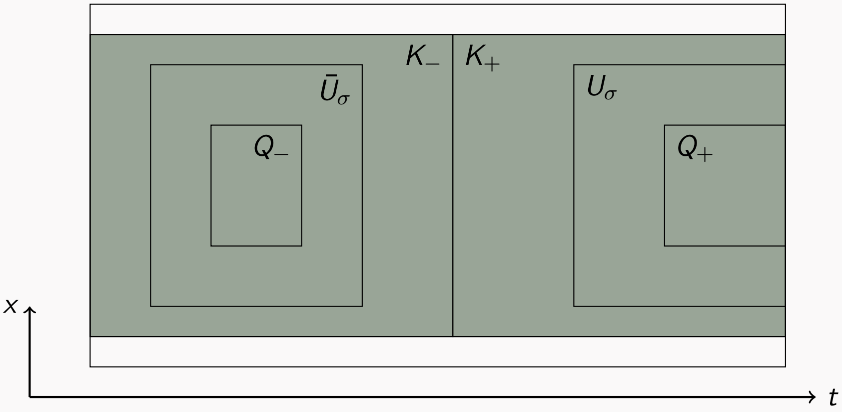

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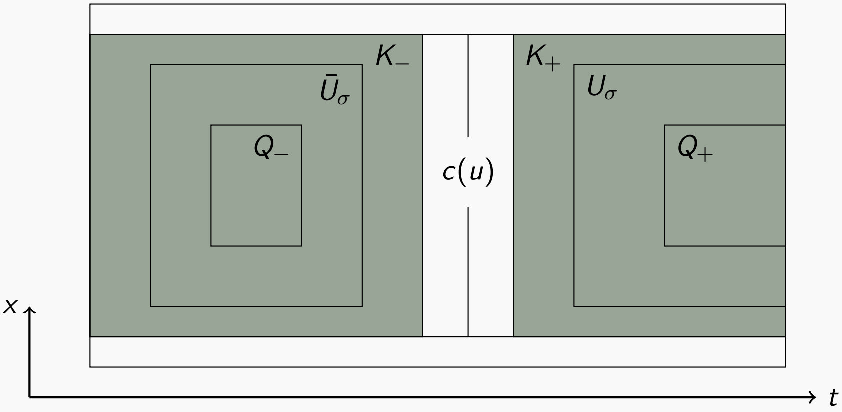

Proof of the weak L^1-estimate mod.

By scaling and translation $t_0 = 0$, $r = 1$. $\tau = 1$ for simplicity.

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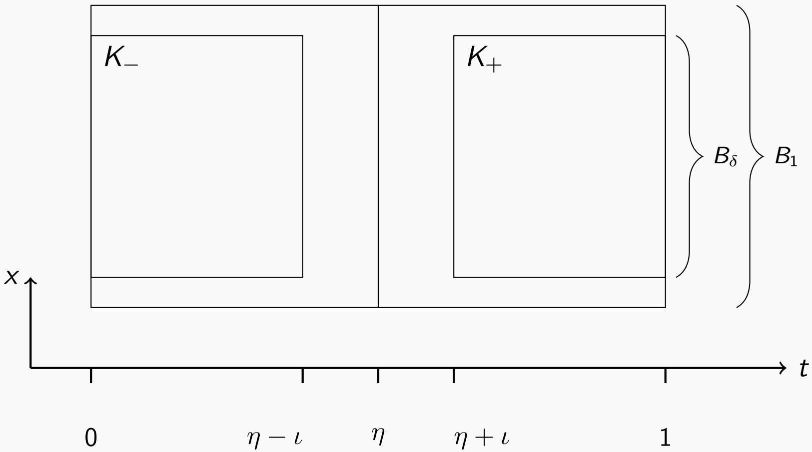

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$$s\,|\{(t,x) \in K_- :\ \log(u) - c(u) > s\}| \leq \int_0^{\eta-\iota} \int_B ([\log u](t,x) - c(u))_+ \mathrm{d}x\mathrm{d}t$$

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Goal: estimate

$$\int\limits_0^{\eta-\iota} \int\limits_B ([\log u](t,x) - c(u))_+ \mathrm{d}x\mathrm{d}t$$

by a constant

L^1-Poincaré inequality in space time without gradient?!

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L^1-Poincaré inequality in space time without gradient?!

If u is solution to (1), then $g = \log u$ is a super solution to

$$\partial_t g = \nabla \cdot (A \nabla g) + \langle A \nabla g, \nabla g \rangle.$$

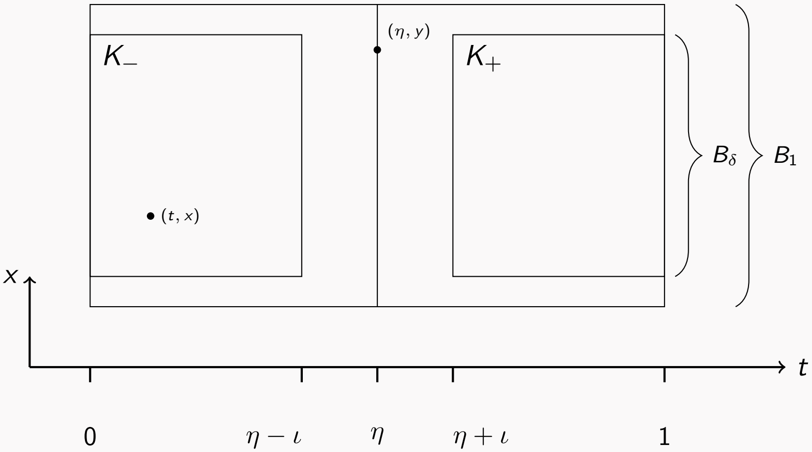
Proof using parabolic trajectories

For $g = \log u$ we have

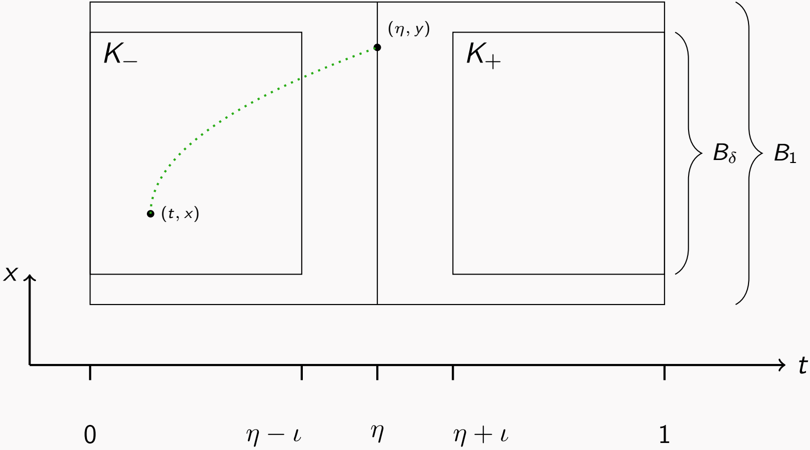
$$g(t,x) - c(u) = \frac{1}{c_\varphi}\int_B (g(t,x) - g(\eta,y))\varphi^2(y)\mathrm{d}y$$
$$= -\frac{1}{c_\varphi}\int_B\int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r))\mathrm{d}r\varphi^2(y)\mathrm{d}y$$

What is a good choice for γ?

Parabolic trajectories



Parabolic trajectories



Proof using parabolic trajectories

For $g = \log u$ we have

$$g(t,x) - c(u) = \frac{1}{c_\varphi}\int_B (g(t,x) - g(\eta,y))\varphi^2(y)\mathrm{d}y$$

$$= -\frac{1}{c_\varphi}\int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r))\mathrm{d}r\varphi^2(y)\mathrm{d}y$$

Parabolic trajectory: $\gamma(r) = (t + r^2(\eta - t), x + r(y - x))$

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$$\leq \frac{1}{c_\varphi}\int_0^1\int_B \Big(-2(\eta - t)r[\nabla\cdot(A\nabla g)](\gamma(r)) - 2(\eta - t)r[\langle A\nabla g, \nabla g\rangle](\gamma(r)) \\ -(y-x)\cdot[\nabla g](\gamma(r))\Big)\varphi^2(y)\mathrm{d}y\mathrm{d}r,$$

Parabolic trajectory: $\gamma(r) = (t + r^2(\eta - t), x + r(y-x))$

Idea: use quadratic gradient term to absorb all gradients

Kinetic equations

Here: $x, v \in \mathbb{R}^n$, $t \in [0, T]$, $u = u(t, x, v)$ particle density

$$\partial_t u + v \cdot \nabla_x u = \nabla_v \cdot (A(t, x, v) \nabla_v u)$$

important prototype in kinetic theory.

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Can Moser's method be applied in the kinetic setting?

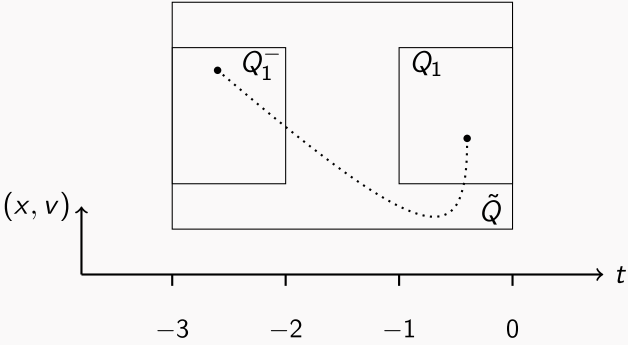
Kinetic Poincaré inequality

Theorem (Guerand & Mouhot 22, N. & Zacher 22):

Let $u \geq 0$ be a subsolution to (1) in $\tilde{Q}$. Then

$$\left\| (u - \langle u\varphi^2 \rangle_{Q_1^-})_+ \right\|_{L^1(Q_1)} \leq C \left\| \nabla_v u \right\|_{L^1(\tilde{Q})}.$$

with φ^2 supported in Q_1^-.



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Kinetic maximal L^p-regularity

- optimal regularity estimates for kinetic equations
- framework to study wellposedness of quasilinear kinetic equations

L. N., R. Zacher, *Kinetic maximal L^2-regularity for the (fractional) Kolmogorv equation*. Journal of Evolution Equations 21 (2021).

L. N., R. Zacher, *Kinetic maximal L^p-regularity with temporal weights and application to quasilinear kinetic diffusion equations*. Journal of Differential Equations 307 (2022).

L. N., Kinetic maximal $L^p_\mu(L^p)$-regularity for the fractional Kolmogorov equation with variable density. Nonlinear Analysis (2022).

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lukasniebel.github.io