



Mollification in kinetic geometry

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Elliptic Sobolev inequality

Theorem:

For $1 < p < d$ and $p^* = \frac{dp}{d-p}$ we have

$$\|f\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

Elliptic geometry

Laplace equation with $f: \mathbb{R}^d \rightarrow \mathbb{R}$:

$$-\Delta f = 0$$

Translation invariance for $h \in \mathbb{R}^d$:

$$x \circ h = x + h \text{ and } \Delta(f(\cdot + h))(x) = [\Delta f](x + h)$$

Scaling invariance for $\lambda > 0$:

$$\delta_\lambda x = \lambda x \text{ and } \Delta(f(\lambda \cdot))(x) = \lambda^2 [\Delta f](\lambda x)$$

Trajectories $\gamma = \gamma(\tau): [0, \infty) \rightarrow \mathbb{R}^d$ are straight lines

For a direction $m = m_1 \in \mathbb{R}^d$ and a starting point $x \in \mathbb{R}^d$:

$$\gamma^m(\tau) = x - \tau m, \quad \gamma^m(0) = x, \quad \dot{\gamma}^m(\tau) = -m$$

Elliptic mollification

$$\gamma^m(\tau) = x - \tau m$$

Choose $\psi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\psi \geq 0, \quad \int_{\mathbb{R}^d} \psi(m) \, dm = 1, \quad \text{supp } \psi \subset \{m \in \mathbb{R}^d : 1 < |m| < 2\}$$

For $\tau > 0$ define the elliptic mollification kernel

$$K_\tau(y) := \tau^{-d} \psi\left(\frac{y}{\tau}\right)$$

Mollification is a trajectory average at scale τ :

$$\begin{aligned} [T_{K_\tau} f](x) &:= \int_{\mathbb{R}^d} f(x - y) K_\tau(y) \, dy \\ &= \int_{\mathbb{R}^d} f(x - \tau m) \psi(m) \, dm \\ &= \int_{\mathbb{R}^d} f(\gamma^m(\tau)) \psi(m) \, dm \end{aligned}$$

Representation formula

$$\gamma^m(r) = x - rm$$

Apply the fundamental theorem of calculus along $r \mapsto \gamma^m(r)$.

$$\begin{aligned} f(x) - [T_{K_\tau} f](x) &= \int_{\mathbb{R}^d} (f(x) - f(x - \tau m)) \psi(m) \, dm \\ &= \int_{\mathbb{R}^d} \int_0^\tau m \cdot \nabla f(x - rm) \, dr \psi(m) \, dm \\ &= \int_{\mathbb{R}^d} \nabla f(x - y) \cdot \int_0^\tau r^{-d} \psi\left(\frac{y}{r}\right) \frac{y}{r} \, dr \, dy. \end{aligned}$$

Kernel

$$\begin{aligned} \mathcal{B}_\tau(y) &:= \int_0^\tau r^{-d} \psi\left(\frac{y}{r}\right) \frac{y}{r} \, dr \\ [T_{\mathcal{B}_\tau} F](x) &:= \int_{\mathbb{R}^d} F(x - y) \cdot \mathcal{B}_\tau(y) \, dy \end{aligned}$$

Representation

$$f = f - T_{K_\tau} f + T_{K_\tau} f = T_{\mathcal{B}_\tau}(\nabla f) + T_{K_\tau} f$$

Young inequalities

For a kernel J set

$$[T_J g](x) = \int_{\mathbb{R}^d} J(x-y) \cdot g(y) dy$$

Young inequality ($1 \leq p, q, \theta \leq \infty$):

$$\|T_J g\|_{L^q} \leq \|J\|_{L^\theta} \|g\|_{L^p}, \quad \frac{1}{q} + 1 = \frac{1}{\theta} + \frac{1}{p}$$

Weak Young inequality ($1 < p, q, \theta < \infty$):

$$\|T_J g\|_{L^q} \lesssim \|J\|_{L^{\theta, \infty}} \|g\|_{L^p}, \quad \frac{1}{q} + 1 = \frac{1}{\theta} + \frac{1}{p}$$

Kernel estimates

$$\mathcal{B}_\tau(y) = \int_0^\tau r^{-d} \psi\left(\frac{y}{r}\right) \frac{y}{r} dr$$

Critical scale $\theta = \frac{d}{d-1}$

(i) Scaling of the mollifier

$$\|K_\tau\|_{L^\theta(\mathbb{R}^d)} = \tau^{d(\frac{1}{\theta}-1)} \|\psi\|_{L^\theta(\mathbb{R}^d)} \leq C\tau^{-1}$$

(change of variables)

(ii) Critical bound for the integrated kernel

$$|\mathcal{B}_\tau(y)| \leq C \int_{|y|/2}^{|y|} r^{-d} dr \leq C |y|^{1-d}, \quad \mathcal{B}_\tau(y) = 0 \text{ for } |y| \geq 2\tau$$

$$\|\mathcal{B}_\tau\|_{L^{\frac{d}{d-1}, \infty}(\mathbb{R}^d)} \leq C$$

($\text{supp } \psi(y/r)$; $\|f\|_{L^{p, \infty}} := \sup_{\lambda > 0} \lambda |\{x \in X : |f(x)| > \lambda\}|^{\frac{1}{p}}$)

Elliptic Sobolev inequality

Theorem:

For $1 < p < d$ and $p^* = \frac{dp}{d-p}$ we have

$$\|f\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

Proof:

Note $\frac{1}{p^*} + 1 = \frac{1}{p} + \frac{1}{d-1}$, and write

$$f = (f - T_{K_\tau} f) + T_{K_\tau} f.$$

Use the (weak) Young inequality:

$$\|f - T_{K_\tau} f\|_{L^{p^*}} = \|T_{\mathcal{B}_\tau}(\nabla f)\|_{L^{p^*}} \leq C \|\mathcal{B}_\tau\|_{L^{\frac{d}{d-1}, \infty}} \|\nabla f\|_{L^p} \leq C \|\nabla f\|_{L^p}$$

$$\|T_{K_\tau} f\|_{L^{p^*}} = \|K_\tau * f\|_{L^{p^*}} \leq \|K_\tau\|_{L^{\frac{d}{d-1}}} \|f\|_{L^p} \leq C \tau^{-1} \|f\|_{L^p}$$

$$\|f\|_{L^{p^*}} \leq C \|\nabla f\|_{L^p} + C \tau^{-1} \|f\|_{L^p}, \quad \tau \rightarrow \infty$$

Parabolic geometry

Parabolic equation on \mathbb{R}^{1+d} :

$$\partial_t f = \Delta_x f$$

Translation invariance for $h = (h_0, h_1) \in \mathbb{R}^{1+d}$:

$$(t, x) \circ h = (t + h_0, x + h_1)$$

Parabolic scaling for $\lambda > 0$:

$$\delta_\lambda(t, x) = (\lambda^2 t, \lambda x), \quad Q = 2 + d$$

Parabolic curves for direction $m = (m_0, m_1) \in \mathbb{R}^{1+d}$ and $\tau \in [0, \infty)$:

$$\gamma^m(\tau; (t, x)) = (t - \tau^2 m_0, x - \tau m_1)$$

$$\frac{d}{d\tau} \gamma^m(\tau; (t, x)) = (-2\tau m_0, -m_1)$$

Parabolic regularity

Parabolic equation on \mathbb{R}^{1+d} :

$$\partial_t f = \nabla_x \cdot (\nabla_x f)$$

Weak (energy) solutions:

$$f \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$$

Parabolic regularity:

$$S \quad \text{in} \quad \partial_t f = \nabla_x \cdot S \quad \text{and} \quad \nabla_x f$$

Parabolic mollification

$$\gamma^m(\tau; (t, x)) = (t - \tau^2 m_0, x - \tau m_1)$$

Choose $\psi \in C_c^\infty(\mathbb{R}^{1+d})$ such that

$$\psi \geq 0, \quad \int_{\mathbb{R}^{1+d}} \psi(m) \, dm = 1, \quad \text{supp } \psi \subset (1, 2) \times B_1(0).$$

Averaging along parabolic trajectories at scale τ :

$$\begin{aligned} [T_{K_\tau} f](t, x) &:= \int_{\mathbb{R}^{1+d}} f(\gamma^m(\tau; (t, x))) \psi(m) \, dm \\ &= \int_{\mathbb{R}^{1+d}} f(t - s, x - y) K_\tau(s, y) \, ds \, dy \end{aligned}$$

where for $\tau > 0$ we define the parabolic mollifier

$$K_\tau(s, y) := \tau^{-(d+2)} \psi\left(\frac{s}{\tau^2}, \frac{y}{\tau}\right).$$

Representation formula I

$$\gamma^m(r; (t, x)) = (t - r^2 m_0, x - r m_1)$$

Apply the fundamental theorem of calculus to $r \mapsto f(\gamma^m(r; (t, x)))$:

$$\begin{aligned} & f(t, x) - T_{K_\tau} f(t, x) \\ &= - \int_{\mathbb{R}^{1+d}} \int_0^\tau \frac{d}{dr} f(\gamma^m(r; (t, x))) \, dr \, \psi(m) \, dm \\ &= \int_0^\tau \int_{\mathbb{R}^{1+d}} \left(2rm_0 \partial_t f + m_1 \cdot \nabla_x f \right) (\gamma^m(r; (t, x))) \, \psi(m) \, dm \, dr \\ &=: I_S(t, x) + I_{\nabla f}(t, x) \end{aligned}$$

Representation formula II

For I_S , insert $\partial_t f = \nabla_x \cdot S$ and set

$$s = r^2 m_0, \quad y = r m_1, \quad dm = r^{-Q} ds dy.$$

$$\begin{aligned} I_S(t, x) &= \int_0^\tau \int_{\mathbb{R}^{1+d}} 2rm_0 \partial_t f(\gamma^m(r; (t, x))) \psi(m) dm dr \\ &= \int_0^\tau \int_{\mathbb{R}^{1+d}} 2rm_0 (\nabla_x \cdot S)(t - r^2 m_0, x - r m_1) \psi(m) dm dr \\ &= \int_0^\tau \int_{\mathbb{R}^{1+d}} 2 \frac{s}{r} r^{-Q} (\nabla_x \cdot S)(t - s, x - y) \psi\left(\frac{s}{r^2}, \frac{y}{r}\right) ds dy dr \\ &= - \int_0^\tau \int_{\mathbb{R}^{1+d}} 2 \frac{s}{r} r^{-Q} (\nabla_y \cdot S)(t - s, x - y) \psi\left(\frac{s}{r^2}, \frac{y}{r}\right) ds dy dr \\ &= \int_{\mathbb{R}^{1+d}} S(t - s, x - y) \cdot 2 \int_0^\tau r^{-Q} \frac{s}{r^2} \nabla_{m_1} \psi\left(\frac{s}{r^2}, \frac{y}{r}\right) dr ds dy \\ &= \int_{\mathbb{R}^{1+d}} S(t - s, x - y) \cdot \mathcal{P}_\tau(s, y) ds dy \end{aligned}$$

Representation formula III

$$\partial_t f = \nabla_x \cdot S$$

Forcing term:

$$\begin{aligned} I_{\nabla f}(t, x) &= \int_0^T \int_{\mathbb{R}^{1+d}} m_1 \cdot \nabla_x f(t - r^2 m_0, x - r m_1) \psi(m) \, dm \, dr \\ &= \int_{\mathbb{R}^{1+d}} \nabla_x f(t - s, x - y) \cdot \left[\int_0^T r^{-Q} \frac{y}{r} \psi\left(\frac{s}{r^2}, \frac{y}{r}\right) \, dr \right] \, ds \, dy \end{aligned}$$

Kernels

$$\mathcal{P}_\tau(s, y) := 2 \int_0^\tau r^{-Q} \frac{s}{r^2} \nabla_{m_1} \psi\left(\frac{s}{r^2}, \frac{y}{r}\right) \, dr$$

$$\mathcal{B}_\tau(s, y) := \int_0^\tau r^{-Q} \frac{y}{r} \psi\left(\frac{s}{r^2}, \frac{y}{r}\right) \, dr$$

Representation

$$f = f - T_{K_\tau} f + T_{K_\tau} f = T_{\mathcal{P}_\tau} S + T_{\mathcal{B}_\tau}(\nabla_x f) + T_{K_\tau} f$$

Kernel estimates

$$\text{Critical scale } \theta = \frac{d+2}{d+1} = \frac{Q}{Q-1}$$

(i) Pointwise bound

$$|\mathcal{P}_\tau(s, y)| + |\mathcal{B}_\tau(s, y)| \leq C \frac{\mathbb{1}_{\{0 < s < 2\tau^2\}} \mathbb{1}_{\{|y| \leq \sqrt{s}\}}}{s^{\frac{d+1}{2}}}$$

(ii) Weak bound

$$\|\mathcal{P}_\tau\|_{L^{\theta, \infty}(\mathbb{R}^{1+d})} + \|\mathcal{B}_\tau\|_{L^{\theta, \infty}(\mathbb{R}^{1+d})} \leq C$$

(iii) Mollifier scaling

$$\|\mathcal{K}_\tau\|_{L^\theta(\mathbb{R}^{1+d})} = C\tau^{-1}$$

(on $\text{supp } \psi(s/r^2, y/r)$ one has $r^2 < s < 2r^2$ and $|y| < r$, hence $r \sim \sqrt{s}$;

(iii) is the parabolic change of variables $(s, y) = (\tau^2\sigma, \tau\eta)$)

Parabolic Sobolev inequality

Theorem:

For $p \in (1, d+2)$ set $p^* = \frac{(d+2)p}{d+2-p}$. All f, S with $\partial_t f = \nabla_x \cdot S$ satisfy

$$\|f\|_{L^{p^*}(\mathbb{R}^{1+d})} \leq C(\|\nabla_x f\|_{L^p(\mathbb{R}^{1+d})} + \|S\|_{L^p(\mathbb{R}^{1+d})}).$$

Proof:

$$f = (f - T_{K_\tau} f) + T_{K_\tau} f, \quad 1 + \frac{1}{p^*} = \frac{1}{p} + \frac{1}{\theta}, \quad \theta = \frac{d+2}{d+1}.$$

$$\begin{aligned} \|f - T_{K_\tau} f\|_{L^{p^*}} &\leq \|T_{\mathcal{P}_\tau} S\|_{L^{p^*}} + \|T_{\mathcal{B}_\tau}(\nabla_x f)\|_{L^{p^*}} \\ &\leq C(\|\mathcal{P}_\tau\|_{L^{\theta, \infty}} \|S\|_{L^p} + \|\mathcal{B}_\tau\|_{L^{\theta, \infty}} \|\nabla_x f\|_{L^p}) \\ &\leq C(\|S\|_{L^p} + \|\nabla_x f\|_{L^p}), \end{aligned}$$

$$\|T_{K_\tau} f\|_{L^{p^*}} \leq \|K_\tau\|_{L^\theta} \|f\|_{L^p} \leq C_\tau^{-1} \|f\|_{L^p}$$

$$\|f\|_{L^{p^*}} \leq C(\|S\|_{L^p} + \|\nabla_x f\|_{L^p}) + C_\tau^{-1} \|f\|_{L^p}, \quad \tau \rightarrow \infty$$

Kinetic Landau equation

For $t \geq 0$, $x \in \Omega \subset \mathbb{R}^d$, $v \in \mathbb{R}^d$, the density $f = f(t, x, v)$ solves

$$(\partial_t + v \cdot \nabla_x) f = Q(f)$$

where

$$Q(f) = \nabla_v \cdot \left(\int_{\mathbb{R}^d} A(v - v_*) \left(f(v_*) \nabla_v f(v) - f(v) \nabla_{v_*} f(v_*) \right) dv_* \right)$$

and for $z \in \mathbb{R}^d$

$$A(z) = a_{d,\gamma} |z|^\gamma (|z|^2 \text{Id}_d - z \otimes z), \quad a_{d,\gamma} > 0, \quad -d \leq \gamma \leq 1.$$

(we dropped the (t, x) dependence of f)

Kinetic regularity

Kolmogorov equation on \mathbb{R}^{1+2d} :

$$(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot (a(t, x, v) \nabla_v f)$$

Weak (energy) solutions

$$f \in L_t^\infty L_{x,v}^2 \cap L_{t,x}^2 H_v^1$$

Kinetic regularity

$$S \quad \text{in} \quad (\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot S \quad \text{and} \quad \nabla_v f$$

Kinetic geometry

Kolmogorov equation on \mathbb{R}^{1+2d} :

$$(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot (\nabla_v f)$$

Translation invariance (kinetic group structure):

$$(t, x, v) \circ (s, y, w) = (t + s, x + y + s v, v + w)$$

Scaling invariance for $\lambda > 0$:

$$\delta_\lambda(t, x, v) = (\lambda^2 t, \lambda^3 x, \lambda v), \quad Q = 4d + 2$$

Kinetic trajectories will be considered later.

Hörmander's commutator observation:

$$[\partial_{v_i}, \partial_t + v \cdot \nabla_x] = \partial_{x_i}, \quad i = 1, \dots, d$$

$$[T_{K_\tau} f](t, x, v) = \int_{\mathbb{R}^{1+2d}} K_\tau((t, x, v)^{-1} \circ m) f(m) \, dm$$

with

$$K_\tau(s, y, w) = \tau^{-Q} \psi\left(\frac{s}{\tau^2}, \frac{y}{\tau^3}, \frac{w}{\tau}\right)$$

where $\psi \in C_c^\infty$, $\psi \geq 0$, $\int \psi = 1$, $\text{supp } \psi \subset (-2, -1) \times B_1(0) \times B_1(0)$

$$(t, x, v)^{-1} \circ m = (m_0 - t, m_1 - x - (m_0 - t)v, m_2 - v)$$

$$[T_{K_\tau} f](t, x, v) = \int_{\mathbb{R}^{1+2d}} K_\tau((t, x, v)^{-1} \circ m) f(m) \, dm$$

with

$$K_\tau(s, y, w) = 2^d \tau^{-Q} \psi \left(\frac{s}{\tau^2}, \mathcal{A}_{\frac{s}{\tau^2}}(\tau)^{-1} \begin{pmatrix} y \\ w \end{pmatrix} \right)$$

where $\psi \in C_c^\infty$, $\psi \geq 0$, $\int \psi = 1$, $\text{supp } \psi \subset (-2, -1) \times B_1(0) \times B_1(0)$

and

$$\mathcal{A}_{m_0}(\tau)^{-1} = \begin{pmatrix} \frac{\sin(\log \tau) - 3 \cos(\log \tau)}{\tau^3} \text{Id}_d & \frac{2m_0 \cos(\log \tau)}{\tau} \text{Id}_d \\ \frac{3 \sin(\log \tau) + \cos(\log \tau)}{m_0 \tau^3} \text{Id}_d & -\frac{2 \sin(\log \tau)}{\tau} \text{Id}_d \end{pmatrix}$$

$$(t, x, v)^{-1} \circ m = (m_0 - t, m_1 - x - (m_0 - t)v, m_2 - v)$$

Kinetic trajectories

Consider $\gamma = (\gamma_t, \gamma_x, \gamma_v): [0, 1] \rightarrow \mathbb{R}^{1+2d}$

$$\begin{aligned}\frac{d}{d\tau}f(\gamma(\tau)) &= \dot{\gamma}_t(\tau)[\partial_t f](\gamma(\tau)) + \dot{\gamma}_x(\tau) \cdot [\nabla_x f](\gamma(\tau)) + \dot{\gamma}_v(\tau) \cdot [\nabla_v f](\gamma(\tau)) \\ &= \dot{\gamma}_t(\tau)[(\partial_t + v \cdot \nabla_x)f](\gamma(\tau)) + \dot{\gamma}_v(\tau) \cdot [\nabla_v f](\gamma(\tau))\end{aligned}$$

We call γ a kinetic trajectory if

$$\dot{\gamma}_x(\tau) = \dot{\gamma}_t(\tau)\gamma_v(\tau)$$

Kinetic trajectories via Newton laws

Construct $\gamma = (\gamma_t, \gamma_x, \gamma_v): [0, \infty) \rightarrow \mathbb{R}^{1+2d}$ with $\gamma(0) = (t, x, v)$ and $\dot{\gamma}_x(\tau) = \dot{\gamma}_t(\tau)\gamma_v(\tau)$.

Ansatz:

$$\gamma_t^m(\tau) = t + m_0\tau^2$$

$$\dot{\gamma}_t^m(\tau) = 2m_0\tau$$

$$\dot{\gamma}_v^m(\tau) = \frac{1}{m_0} \partial_\tau \left(\frac{\dot{g}_1(\tau)}{2\tau} \right) m_1 + \partial_\tau \left(\frac{\dot{g}_2(\tau)}{2\tau} \right) m_2$$

for $m = (m_0, m_1, m_2) \in \mathbb{R}^{1+2d}$ and $g_1, g_2: [0, \infty) \rightarrow \mathbb{R}$

Solving Newton's laws gives

$$\gamma_v^m(\tau) = v + \frac{\dot{g}_1(\tau)}{2m_0\tau} m_1 + \frac{\dot{g}_2(\tau)}{2\tau} m_2$$

$$\gamma_x^m(\tau) = x + m_0\tau^2 v + g_1(\tau)m_1 + m_0g_2(\tau)m_2$$

Kinetic trajectories via Newton laws

Let $m = (m_0, m_1, m_2) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^d \times \mathbb{R}^d$
and $g_1, g_2: [0, \infty) \rightarrow \mathbb{R}$ then

$$\gamma^m(\tau) = \left(\begin{array}{c} t + m_0\tau^2 \\ \underbrace{\begin{pmatrix} g_1(\tau) \text{Id}_d & m_0 g_2(\tau) \text{Id}_d \\ \frac{\dot{g}_1(\tau)}{2m_0\tau} \text{Id}_d & \frac{\dot{g}_2(\tau)}{2\tau} \text{Id}_d \end{pmatrix}}_{\mathcal{A}_{m_0}(\tau)} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} \text{Id}_d & m_0\tau^2 \text{Id}_d \\ 0_d & \text{Id}_d \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \end{array} \right)$$

is such that $\gamma^m(0) = (t, x, v)$ and $\dot{\gamma}_x^m(\tau) = \dot{\gamma}_t^m(\tau)\gamma_v^m(\tau)$.

Choice of the forcings

Let $m = (m_0, m_1, m_2) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^d \times \mathbb{R}^d$ and choose

$$g_1(\tau) = \tau^3 \sin(\log \tau), \quad g_2(\tau) = \tau^3 \cos(\log \tau) \quad (\tau > 0).$$

$$\gamma^m(\tau) = \left(\begin{array}{c} t + m_0 \tau^2 \\ \left(\begin{array}{cc} \tau^3 \sin(\log \tau) \text{ Id}_d & m_0 \tau^3 \cos(\log \tau) \text{ Id}_d \\ \frac{\tau}{2m_0} (3 \sin(\log \tau) + \cos(\log \tau)) \text{ Id}_d & \frac{\tau}{2} (3 \cos(\log \tau) - \sin(\log \tau)) \text{ Id}_d \end{array} \right) \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \dots \end{array} \right)$$

is such that $\gamma^m(0) = (t, x, v)$ and $\dot{\gamma}_x^m(\tau) = \dot{\gamma}_t^m(\tau) \gamma_v^m(\tau)$.

Mollification as averaging along trajectories

Rewrite

$$\gamma^m(\tau; (t, x, v)) = (t, x, v) \circ \left(m_0 \tau^2, \mathcal{A}_{m_0}(\tau) \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \right)$$

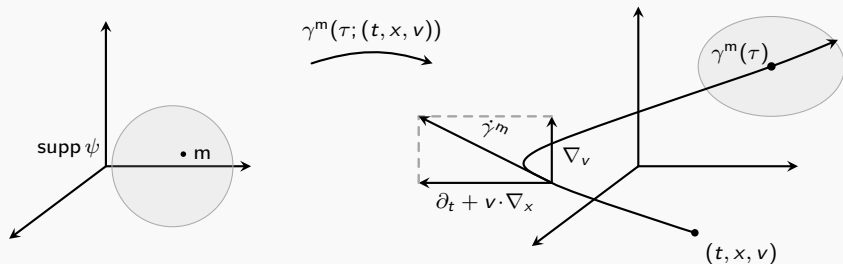
Recall

$$K_\tau(s, y, w) = 2^d \tau^{-Q} \psi \left(\frac{s}{\tau^2}, \mathcal{A}_{\frac{s}{\tau^2}}(\tau)^{-1} \begin{pmatrix} y \\ w \end{pmatrix} \right), \quad Q = 4d + 2$$

$$\begin{aligned} [T_{K_\tau} f](t, x, v) &= \int_{\mathbb{R}^{1+2d}} K_\tau((t, x, v)^{-1} \circ (s, y, w)) f(s, y, w) \, d(s, y, w) \\ &= \int_{\mathbb{R}^{1+2d}} f(\gamma^m(\tau; (t, x, v))) \psi(m) \, dm \end{aligned}$$

Mollification as averaging along trajectories

$$[T_{K_\tau} f](t, x, v) = \int_{\mathbb{R}^{1+2d}} f(\gamma^m(\tau; (t, x, v))) \psi(m) dm.$$



Kinetic trajectories are controllable

Let (t_0, x_0, v_0) and (t_1, x_1, v_1) with $t_1 \neq t_0$.

Set

$$m_0 = t_1 - t_0.$$

Since the endpoint matrix is invertible,

$$\det \mathcal{A}_{m_0}(1) = (-2)^{-d} \neq 0,$$

choose the remaining parameters by solving the endpoint equation:

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \mathcal{A}_{m_0}(1)^{-1} \left[\begin{pmatrix} x_1 \\ v_1 \end{pmatrix} - \begin{pmatrix} \text{Id}_d & m_0 \text{Id}_d \\ 0_d & \text{Id}_d \end{pmatrix} \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \right]$$

$$\gamma^m(1; (t_0, x_0, v_0)) = (t_1, x_1, v_1)$$

$$[T_{K_\tau} f](t, x, v) = \int_{\mathbb{R}^{1+2d}} K_\tau((t, x, v)^{-1} \circ m) f(m) \, dm$$

with

$$K_\tau(s, y, w) = 2^d \tau^{-Q} \psi \left(\frac{s}{\tau^2}, \mathcal{A}_{\frac{s}{\tau^2}}(\tau)^{-1} \begin{pmatrix} y \\ w \end{pmatrix} \right)$$

where $\psi \in C_c^\infty$, $\psi \geq 0$, $\int \psi = 1$, $\text{supp } \psi \subset (-2, -1) \times B_1(0) \times B_1(0)$

and

$$\mathcal{A}_{m_0}(\tau)^{-1} = \begin{pmatrix} \frac{\sin(\log \tau) - 3 \cos(\log \tau)}{\tau^3} \text{Id}_d & \frac{2m_0 \cos(\log \tau)}{\tau} \text{Id}_d \\ \frac{3 \sin(\log \tau) + \cos(\log \tau)}{m_0 \tau^3} \text{Id}_d & -\frac{2 \sin(\log \tau)}{\tau} \text{Id}_d \end{pmatrix}$$

$$(t, x, v)^{-1} \circ m = (m_0 - t, m_1 - x - (m_0 - t)v, m_2 - v)$$

$$[T_{K_\tau} f](t, x, v) = \int_{\mathbb{R}^{1+2d}} K_\tau((t, x, v)^{-1} \circ m) f(m) \, dm$$

with

$$K_\tau(s, y, w) = 2^d \tau^{-Q} \psi \left(\frac{s}{\tau^2}, \mathcal{A}_{\frac{s}{\tau^2}}(\tau)^{-1} \begin{pmatrix} y \\ w \end{pmatrix} \right)$$

where $\psi \in C_c^\infty$, $\psi \geq 0$, $\int \psi = 1$, $\text{supp } \psi \subset (-2, -1) \times B_1(0) \times B_1(0)$

and $\mathcal{A}_{m_0}(\tau): [0, \infty) \rightarrow \mathbb{R}^{2d \times 2d}$, $m_0 \neq 0$ with

$$\text{(C1)} \quad \det \mathcal{A}_{m_0}(\tau) \sim \tau^{4d}$$

$$\text{(C2)} \quad |(\mathcal{A}_{m_0}(\tau)^{-1})_{\cdot;2}| \lesssim_{m_0} \tau^{-1}$$

$$\text{(C3)} \quad |\partial_\tau (\mathcal{A}_{m_0}(\tau))_{2;\cdot}| \lesssim_{m_0} 1$$

Representation formula I

$$Q = 4d + 2$$

Assume $(\partial_t + v \cdot \nabla_x)f = \nabla_v \cdot S$. Along kinetic trajectories:

$$\begin{aligned} \frac{d}{dr} f(\gamma^m(r)) &= 2rm_0 [(\partial_t + v \cdot \nabla_x)f](\gamma^m(r)) + \dot{\gamma}_v^m(r) \cdot [\nabla_v f](\gamma^m(r)) \\ &= 2rm_0 [\nabla_v \cdot S](\gamma^m(r)) + \dot{\gamma}_v^m(r) \cdot [\nabla_v f](\gamma^m(r)) \end{aligned}$$

Apply the fundamental theorem of calculus to $r \mapsto f(\gamma^m(r))$:

$$\begin{aligned} f(t, x, v) &- T_{K_\tau} f(t, x, v) \\ &= - \int_{\mathbb{R}^{1+2d}} \int_0^\tau \frac{d}{dr} f(\gamma^m(r)) dr \psi(m) dm \\ &= - \int_0^\tau \int_{\mathbb{R}^{1+2d}} \left(2rm_0 [\nabla_v \cdot S] + \dot{\gamma}_v^m(r) \cdot \nabla_v f \right) (\gamma^m(r)) \psi(m) dm dr \\ &=: I_S(t, x, v) + I_\nabla f(t, x, v) \end{aligned}$$

Representation formula II

Change of variables

$$(s, y, w) = \left(m_0 r^2, \mathcal{A}_{m_0}(r) \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \right), \quad \gamma^m(r) = (t, x, v) \circ (s, y, w)$$

$$m := \left(\frac{s}{r^2}, \mathcal{A}_{\frac{s}{r^2}}(r)^{-1} \begin{pmatrix} y \\ w \end{pmatrix} \right), \quad dm = 2^d r^{-Q} d(s, y, w)$$

$I_S(t, x, v)$

$$= - \int_0^T \int_{\mathbb{R}^{1+2d}} 2^{d+1} s r^{-Q-1} [\nabla_v \cdot S]((t, x, v) \circ (s, y, w)) \psi(m(s, y, w)) d(s, y, w) dr$$

$$= - \int_0^T \int_{\mathbb{R}^{1+2d}} 2^{d+1} s r^{-Q-1} \nabla_w \cdot [S((t, x, v) \circ (s, y, w))] \psi(m(s, y, w)) d(s, y, w) dr$$

$$= \int_{\mathbb{R}^{1+2d}} S((t, x, v) \circ (s, y, w)) \cdot \mathcal{P}_\tau(s, y, w) d(s, y, w).$$

$$\mathcal{P}_\tau(s, y, w) = 2^{d+1} \int_0^\tau r^{-Q} \frac{s}{r^2} [\nabla_{(m_1, m_2)} \psi]^T \left(\frac{s}{r^2}, \mathcal{A}_{\frac{s}{r^2}}(r)^{-1} \begin{pmatrix} y \\ w \end{pmatrix} \right) \left(\mathcal{A}_{\frac{s}{r^2}}(r)^{-1} \right)_{:,2} r dr$$

Representation formula III

$$\dot{\gamma}_v^m(r) = \partial_r(\mathcal{A}_{m_0}(r))_{2;\cdot} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

Same change of variables

$$I_{\nabla f}(t, x, v) = \int_{\mathbb{R}^{1+2d}} [\nabla_v f]((t, x, v) \circ (s, y, w)) \cdot \mathcal{B}_\tau(s, y, w) d(s, y, w),$$

$$\begin{aligned} \mathcal{B}_\tau(s, y, w) := & -2^d \int_0^\tau r^{-Q} \psi\left(\frac{s}{r^2}, \mathcal{A}_{\frac{s}{r^2}}(r)^{-1} \begin{pmatrix} y \\ w \end{pmatrix}\right) \\ & \times \left[\partial_\rho(\mathcal{A}_{\frac{s}{r^2}}(\rho))_{2;\cdot} \Big|_{\rho=r} \mathcal{A}_{\frac{s}{r^2}}(r)^{-1} \begin{pmatrix} y \\ w \end{pmatrix} \right] dr. \end{aligned}$$

Kernel estimates

$$\text{Critical scale } \theta = \frac{4d+2}{4d+1} = \frac{Q}{Q-1}, \quad Q = 4d + 2$$

(i) Pointwise bound

$$|\mathcal{P}_\tau(s, y, w)| + |\mathcal{B}_\tau(s, y, w)| \lesssim \frac{\mathbb{1}_{\{-2\tau^2 < s < 0\}} \mathbb{1}_{\{|y| \lesssim |s|^{3/2}\}} \mathbb{1}_{\{|w| \lesssim |s|^{1/2}\}}}{|s|^{\frac{Q-1}{2}}}$$

(on the support of the integrand one has $\sqrt{|s|/2} \leq r \leq \sqrt{|s|}$, $|y| \lesssim r^3$, $|w| \lesssim r$; since the r -integrand is $\lesssim r^{-Q}$, integrating in r gives $|s|^{-(Q-1)/2}$)

(ii) Weak bound

$$\|\mathcal{P}_\tau\|_{L^{\theta, \infty}} + \|\mathcal{B}_\tau\|_{L^{\theta, \infty}} \leq C$$

(iii) Mollifier scaling

$$\|K_\tau\|_{L^\theta} = C\tau^{-1}$$

Young inequalities for kinetic convolution

Recall

$$\begin{aligned} [T_J g](t, x, v) &= \int_{\mathbb{R}^{1+2d}} J((t, x, v)^{-1} \circ m) g(m) dm \\ &= \int_{\mathbb{R}^{1+2d}} J(s, y, w) g((t, x, v) \circ (s, y, w)) d(s, y, w) \end{aligned}$$

The kinetic shifts are measure preserving.

Young inequality ($1 \leq p, q, \theta \leq \infty$):

$$\|T_J g\|_{L^q} \leq \|J\|_{L^\theta} \|g\|_{L^p}, \quad \frac{1}{q} + 1 = \frac{1}{\theta} + \frac{1}{p}$$

Weak Young inequality ($1 < p, q, \theta < \infty$):

$$\|T_J g\|_{L^q} \lesssim \|J\|_{L^{\theta, \infty}} \|g\|_{L^p}, \quad \frac{1}{q} + 1 = \frac{1}{\theta} + \frac{1}{p}$$

Kinetic Sobolev inequality: proof

Let $1 < p < Q$, $p^* = \frac{pQ}{Q-p}$, and $\theta = \frac{Q}{Q-1}$.

Representation:

$$f = f - T_{K_\tau} f + T_{K_\tau} f = T_{\mathcal{P}_\tau}(S) + T_{\mathcal{B}_\tau}(\nabla_v f) + T_{K_\tau} f,$$

Kernel estimates:

$$\|\mathcal{P}_\tau\|_{L^{\theta,\infty}} + \|\mathcal{B}_\tau\|_{L^{\theta,\infty}} \lesssim 1.$$

Weak Young inequality:

$$\|f - T_{K_\tau} f\|_{L^{p^*}} \lesssim \|S\|_{L^p} + \|\nabla_v f\|_{L^p}.$$

Mollified part as $\tau \rightarrow \infty$:

$$\|T_{K_\tau} f\|_{L^{p^*}} \leq \|K_\tau\|_{L^\theta} \|f\|_{L^p} \lesssim \tau^{-1} \|f\|_{L^p} \longrightarrow 0.$$

Kinetic Sobolev inequality

Theorem (Dietert–Mouhot–N.-Zacher '25)¹

$$Q = 4d + 2, \quad 1 < p < Q, \quad p^* = \frac{pQ}{Q - p}.$$

Let $f \in L^p(\mathbb{R}^{1+2d})$ with $\nabla_v f \in L^p(\mathbb{R}^{1+2d})$ and suppose

$$(\partial_t + v \cdot \nabla_x)f = \nabla_v \cdot S,$$

with $S \in L^p(\mathbb{R}^{1+2d}; \mathbb{R}^d)$. Then

$$\|f\|_{L^{p^*}(\mathbb{R}^{1+2d})} \lesssim \|\nabla_v f\|_{L^p(\mathbb{R}^{1+2d})} + \|S\|_{L^p(\mathbb{R}^{1+2d})}.$$

¹Earlier versions and related works: Hörmander '67, Pascucci–Polidoro '04, Golse–Imbert–Mouhot–Vasseur '19, Guerand–Mouhot '22, Auscher–Imbert–N. '24, Auscher–N. '26.

Transfer of regularity: Hörmander

Hörmander's Besov seminorm in the x -variable:

$$\|f\|_{L_{t,v}^p \dot{B}_{p,\infty,x}^{1/3}} := \sup_{h \neq 0} \frac{\|\Delta_x^h f\|_{L_{t,x,v}^p}}{|h|^{1/3}}.$$

Theorem (N. '26)¹

$$Q = 4d + 2, \quad p \in \left[\frac{Q}{Q-1}, \infty \right), \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{Q}.$$

Let $f \in \mathcal{S}(\mathbb{R}^{1+2d}) \cap L^p$ satisfy

$$(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot S_0 + S_1,$$

with $\nabla_v f, S_0 \in L^p$ and $S_1 \in L^q$. Then

$$\|f\|_{L_{t,v}^p \dot{B}_{p,\infty,x}^{1/3}} \lesssim \|\nabla_v f\|_{L^p} + \|S_0\|_{L^p} + \|S_1\|_{L^q}.$$

¹Related works and earlier versions: Hörmander '67, Albritton–Armstrong–Mourrat–Novack '24.

Transfer of regularity: Bouchut

Theorem (Auscher-N. '26, N. '26)¹

$$Q = 4d + 2, \quad p \in \left(\frac{Q}{Q-1}, \infty \right), \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{Q}.$$

Let $f \in \mathcal{S}(\mathbb{R}^{1+2d}) \cap L^p$ satisfy

$$(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot S_0 + S_1,$$

with $\nabla_v f, S_0 \in L^p$ and $S_1 \in L^q$. Then

$$\|D_x^{1/3} f\|_{L^p_{t,x,v}} \lesssim \|\nabla_v f\|_{L^p_{t,x,v}} + \|S_0\|_{L^p_{t,x,v}} + \|S_1\|_{L^q_{t,x,v}}.$$

¹Related works and earlier versions: Bouchut '02, N.-Zacher '21, Auscher-Imbert-N. '24.

Kinetic Sobolev inequality: nonlocal diffusion I

Bessel-type control:

$$\widehat{D_v^s g}(\xi) = |\xi|^{2s} \widehat{g}(\xi), \quad 0 < s < 1.$$

$$Q = 2s + 2d(s + 1), \quad 1 < p < \frac{Q}{s}, \quad \frac{1}{p_s^*} = \frac{1}{p} - \frac{s}{Q}.$$

Theorem (Auscher-N. '26, N. '26)¹

Assume

$$(\partial_t + v \cdot \nabla_x) f = D_v^s S, \quad D_v^s f, S \in L^p(\mathbb{R}^{1+2d}).$$

Then the endpoint Sobolev estimate holds:

$$\|f\|_{L^{p_s^*}} \lesssim \|D_v^s f\|_{L^p} + \|S\|_{L^p}.$$

¹Related works and earlier versions: Imbert–Silvestre '20, Pascucci–Pesce '24, Auscher–Imbert–N. '24., Anneschi–Palatucci–Piccinini '24

Kinetic Sobolev inequality: nonlocal diffusion II

Gagliardo-type fractional control:

$$\mathfrak{D}_v^s f(t, x, v, h) = \frac{f(t, x, v + h) - f(t, x, v)}{|h|^s}, \quad d\eta(h) = \frac{dh}{|h|^d}$$
$$Q = 2s + 2d(s + 1), \quad 1 < p < \frac{Q}{s}, \quad \frac{1}{p_s^*} = \frac{1}{p} - \frac{s}{Q}$$

Theorem (N. '26)

Assume

$$(\partial_t + v \cdot \nabla_x) f = \mathfrak{D}_v^{s,*} S, \quad \mathfrak{D}_v^s f, S \in L^p(d\eta \otimes d(t, x, v))$$

Then for every $q \in (p, p_s^*)$,

$$\|f\|_{L^q} \lesssim \|f\|_{L^p}^{1-\vartheta_{p,q}} \left(\|\mathfrak{D}_v^s f\|_{L^p(d\eta \otimes d(t,x,v))} + \|S\|_{L^p(d\eta \otimes d(t,x,v))} \right)^{\vartheta_{p,q}}$$

$$\vartheta_{p,q} = \frac{\frac{1}{p} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{p_s^*}}$$

Kinetic Harnack inequality

Kolmogorov equation with rough diffusion matrix

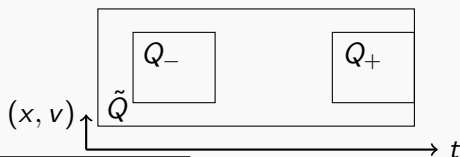
$$(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot (a(t, x, v) \nabla_v f)$$

$$0 < \lambda := \inf_{\substack{(t,x,v) \\ \xi \neq 0}} \frac{\langle a(t, x, v) \xi, \xi \rangle}{|\xi|^2} \quad \Lambda := \sup_{\substack{(t,x,v) \\ \xi \neq 0}} \frac{|a(t, x, v) \xi|^2}{\langle a(t, x, v) \xi, \xi \rangle} < \infty$$

Theorem (Dietert–Mouhot–N.–Zacher '25)¹

There exists a universal constant $C = C(d) > 0$ such that every nonnegative weak solution f in \tilde{Q} satisfies

$$\sup_{Q_-} f \leq C^{\frac{1}{\lambda} + \Lambda} \inf_{Q_+} f.$$



¹Related works and earlier versions: Golse–Imbert–Mouhot–Vasseur '19, Guerland–Imbert '22, Guerland–Mouhot '22

Weak L^1 -estimate for $\log f$

$$(1) \quad \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (a \nabla_v f)$$

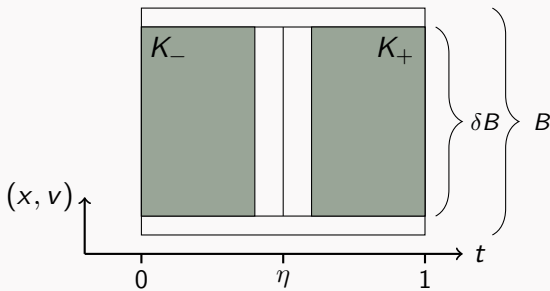
Theorem (Dietert–Mouhot–N.–Zacher '25):

Let $\delta, \eta \in (0, 1)$. Then for any supersolution $f > 0$ to (1) there exists a constant $C = C(d, \delta, \eta, \lambda, \Lambda) > 0$ such that

$$s |\{(t, x, v) \in K_- : \log f(t, x, v) - c(f) > s\}| \leq C$$

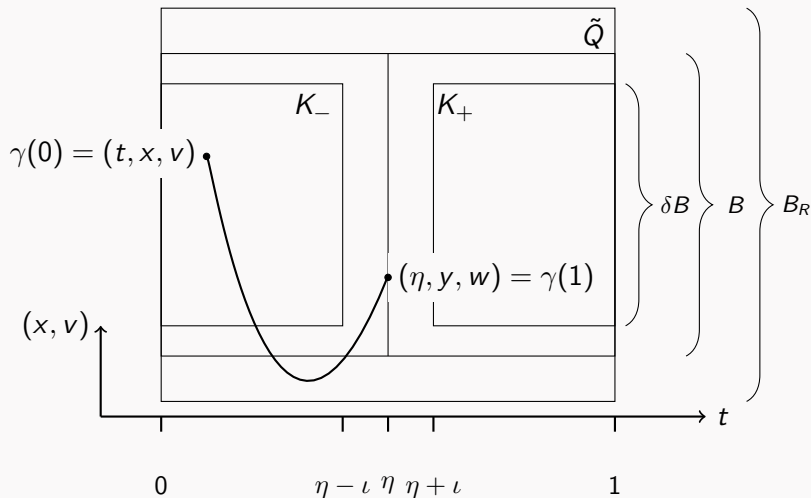
$$s |\{(t, x, v) \in K_+ : c(f) - \log f(t, x, v) > s\}| \leq C$$

for all $s > 0$ with $c(f) = \frac{1}{c_\varphi} \int_B \log f(\eta, y, w) \varphi^2(y, w) d(y, w)$.









Kinetic trajectories and the logarithm

$$(1) \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\alpha \nabla_v f)$$



$$(\partial_t + v \cdot \nabla_x) \log f \geq \nabla_v \cdot (\alpha \nabla_v \log f) + \langle \alpha \nabla_v \log f, \nabla_v \log f \rangle.$$

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