Steady bubbles and drops in inviscid fluids

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Seminar of the AG Nichtlineare Partielle Differentialgleichungen

KIT Karlsruhe, 21st May 2025

1. Experiments (videos, links on the last slide)

2. Two-phase Euler equations with surface tension

3. Travelling wave solutions and an overdetermined elliptic free boundary value problem

4. Close-to-spherical solutions (theorem, remarks, and proof)

Euler equations

Velocity field of the fluid $U = U(t,x)\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ solution to

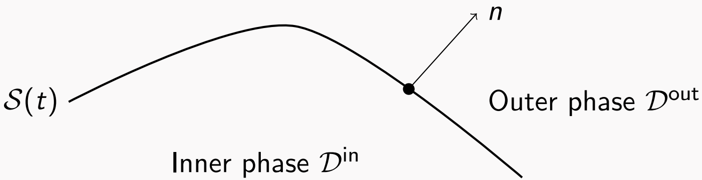
$$\begin{aligned} \partial_t U + (U\cdot\nabla)U + \nabla P &= 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^3 \\ \nabla\cdot U &= 0 \qquad \text{in } \mathbb{R}^3 \end{aligned}$$

where $P\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ is the pressure.

Two-phase Euler equations

Velocity field of the fluid $U\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ solution to

$$\begin{aligned} \rho(\partial_t U + (U\cdot\nabla)U) + \nabla P &= 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^3 \\ \nabla\cdot U &= 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^3 \\ [\![U\cdot n]\!] &= 0 \qquad \text{on } \mathcal{S}(t) \end{aligned}$$



Two-phase Euler equations

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$$\begin{aligned} \rho(\partial_t U + (U \cdot \nabla)U) + \nabla P = 0 \qquad & \text{in } \mathbb{R} \times \mathbb{R}^3 \\ \nabla \cdot U = 0 \qquad & \text{in } \mathbb{R} \times \mathbb{R}^3 \\ [\![U \cdot n]\!] = 0 \qquad & \text{on } \mathcal{S}(t) \end{aligned}$$

where

- $P\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is the pressure
- $\mathcal{S}(t)$ is the interface separating the inner $\mathcal{D}^{\mathrm{in}}(t)$ and outer $\mathcal{D}^{\mathrm{out}}(t)$ fluid domain
- $\rho(t) = \rho^{\mathrm{in}} \mathbb{1}_{\mathcal{D}^{\mathrm{in}}(t)} + \rho^{\mathrm{out}} \mathbb{1}_{\mathcal{D}^{\mathrm{out}}(t)}$ for $\rho^{\mathrm{in}}, \rho^{\mathrm{out}} \geq 0$, is the density function
- $[\![f]\!] = f^{\mathrm{out}} - f^{\mathrm{in}}$, the jump of a quantity f across the interface.

Two-phase Euler equations

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- $\rho(t) = \rho^{\mathrm{in}}\mathbb{1}_{\mathcal{D}^{\mathrm{in}}(t)} + \rho^{\mathrm{out}}\mathbb{1}_{\mathcal{D}^{\mathrm{out}}(t)}$ for $\rho^{\mathrm{in}},\rho^{\mathrm{out}} \geq 0$, is the density function.

Ill-posed due to Kelvin-Helmholtz instability!

Two-phase Euler equations with surface tension

Velocity field of the fluid $U\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ solution to

$$
\begin{aligned}
\rho(\partial_t U + (U \cdot \nabla)U) + \nabla P &= 0 && \text{in } \mathbb{R} \times \mathbb{R}^3 \\
\nabla \cdot U &= 0 && \text{in } \mathbb{R} \times \mathbb{R}^3 \\
[\![P]\!] &= \sigma H && \text{on } \mathcal{S}(t) \\
[\![U \cdot n]\!] &= 0 && \text{on } \mathcal{S}(t)
\end{aligned}
$$

where

- we take into consideration the Young-Laplace law

- H is the mean curvature ($H = 2$ for the unit ball)

- $\sigma > 0$ is the surface tension

Two-phase Euler equations with surface tension

Velocity field of the fluid $U\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ solution to

$$\begin{aligned} \rho(\partial_t U + (U \cdot \nabla)U) + \nabla P &= 0 && \text{in } \mathbb{R} \times \mathbb{R}^3 \\ \nabla \cdot U &= 0 && \text{in } \mathbb{R} \times \mathbb{R}^3 \\ [\![P]\!] &= \sigma H && \text{on } \mathcal{S}(t) \\ [\![U \cdot n]\!] &= 0 && \text{on } \mathcal{S}(t) \end{aligned}$$

Literature:

- Lots of physics literature. Influential: Hou-Lowengrub-Shelly '97
- Locally well-posed: Iguchi-Tanaka-Tani '97, Ambrose '02, Schweizer '05, Ambrose-Masmoudi '2007, Cheng-Coutand-Shkoller '08, Coutand-Shkoller '08
- A priori regularity: Shatah-Zeng '08
- Finite-time singularities: Coutand-Shkoller '14, Castro-Córdoba-Fefferman-Gancedo-Gómez-Serrano '12

Traveling wave solutions

We make the ansatz

$$\begin{aligned} u(x) &= U(t, x_1, x_2, x_3 + Vt) - Ve_3 \\ p(x) &= P(t, x_1, x_2, x_3 + Vt) \\ \mathcal{S}(t) &= \mathcal{S} + tVe_3, \end{aligned}$$

for some speed $V \geq 0$.

Traveling wave solutions

The time-independent $u, p, \mathcal{S}$ solve the steady two-phase Euler equations

$$\begin{aligned}
\rho\,(u \cdot \nabla) u + \nabla p &= 0 && \text{in } \mathbb{R}^3 \backslash \mathcal{S}, \\
\nabla \cdot u &= 0 && \text{in } \mathbb{R}^3, \\
[\![p]\!] &= \sigma H && \text{on } \mathcal{S}, \\
u \cdot n &= 0 && \text{on } \mathcal{S}.
\end{aligned}$$

with $\lim\limits_{|x| \to \infty} u(x) = -V e_3$.

Traveling wave solutions

Bernoulli equations (for steady flows) for the inner/outer phase are

$$\frac{\rho^{\mathrm{in}}}{2}\left|u^{\mathrm{in}}\right|^2 + p^{\mathrm{in}} = \mathrm{const},$$
$$\frac{\rho^{\mathrm{out}}}{2}\left|u^{\mathrm{out}}\right|^2 + p^{\mathrm{out}} = \mathrm{const}.$$

We rewrite the interfacial condition

$$[\![P]\!] = \sigma H \ \text{ on } \mathcal{S}$$

Rewrite the interfacial condition as

$$\frac{1}{2}[\![\rho|u|^2]\!] + \sigma H = \mathrm{const} \ \text{ on } \mathcal{S}.$$

Traveling wave solutions

We are interested in axisymmetric and swirl-free vector fields ($u = u(r, z)$ and azimutal component $u_\varphi = 0$).

We assume uniform vorticity distribution in the inner phase, i.e.

$$\operatorname{curl} u^{\text{in}} = \omega_a = \frac{15}{2} a \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} = \frac{15}{2} a \, r e_\varphi$$

for $a \in \mathbb{R}$.

The fluid in the outer domain is assumed to be irrotational $\operatorname{curl} u^{\text{out}} = 0$.

The volume is $|\mathcal{S}| = \dfrac{4}{3}\pi R^3$.

Traveling wave solutions

We work with the vector stream function $\psi\colon \mathbb{R}^3 \to \mathbb{R}^3$ with

$$u = \operatorname{curl} \psi - V e_3.$$

The tangential flow and the axisymmetry no-swirl condition yields

$$\psi = \frac{V}{2} r e_\varphi \text{ on } \mathcal{S}.$$

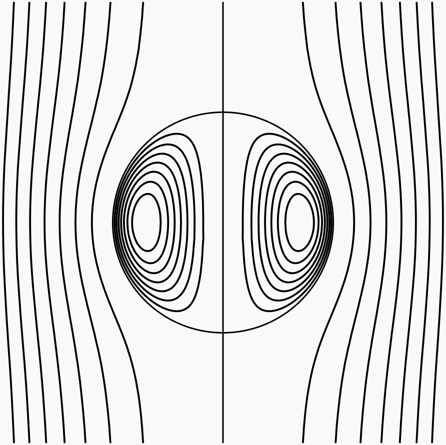
The identity $\operatorname{curl}\operatorname{curl} = \nabla\nabla \cdot - \Delta$ implies

$$-\Delta \psi = \omega_a \mathbb{1}_{\mathcal{D}^{\text{in}}} \text{ in } \mathbb{R}^3 \setminus \mathcal{S}.$$

The jump condition becomes

$$\frac{1}{2} \left[\!\!\left[\rho \left| \operatorname{curl} \psi - V e_3 \right|^2 \right]\!\!\right] + \sigma H = \mathrm{const} \text{ on } \mathcal{S}.$$

Spherical solution with Hill's vortex core



Streamlines of ψ_S in axisymmetric coordinates

Spherical solution with Hill's vortex core

A first solution is given by $\mathcal{S}$ the sphere of radius R

$$\psi_S(x) = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \cdot \begin{cases} \frac{3a}{4}\left(R^2 - |x|^2\right) + \frac{V_S}{2} & \text{for } |x| \leq R \\ \frac{V_S}{2}\frac{R^3}{|x|^3} & \text{for } |x| > R, \end{cases}$$

where $V_S = |a|\, R^2 \sqrt{\frac{\rho^{\text{in}}}{\rho^{\text{out}}}}$ is determined such that

$$\frac{1}{2}[\![\rho |\mathrm{curl}\, \psi_S - V_S e_3|^2]\!] = \frac{9}{8R^2}\left(a^2 R^4 \rho^{\text{in}} - \rho^{\text{out}} V_S^2\right)\left(x_1^2 + x_2^2\right)$$

is constant on the sphere of radius R and thus

$$\frac{1}{2}[\![\rho |\mathrm{curl}\, \psi_S - V_S e_3|^2]\!] + \sigma H = 2\sigma R = \text{const.}$$

Spherical solution with Hill's vortex core

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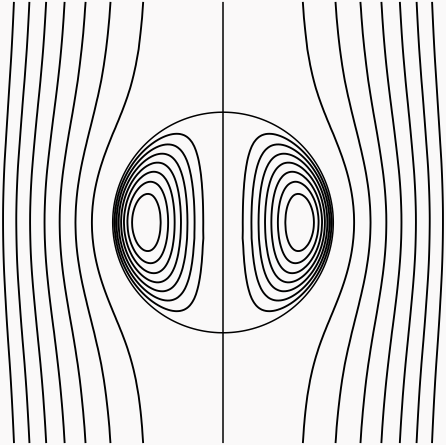
$$\psi_S(x) = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \cdot \begin{cases} \dfrac{3a}{4}\left(R^2 - |x|^2\right) + \dfrac{V_S}{2} & \text{for } |x| \leq R \\ \dfrac{V_S}{2}\dfrac{R^3}{|x|^3} & \text{for } |x| > R, \end{cases}$$

with $V_S = |a|\, R^2 \sqrt{\frac{\rho^{\text{in}}}{\rho^{\text{out}}}}$.

Vortex sheet, i.e. nonzero jump of $U_S \cdot \tau$ at $\mathcal{S}$, whenever $V_S \neq aR^2$.

Spherical solution with Hill's vortex core



Streamlines of ψ_S in axisymmetric coordinates

The overdetermined free boundary value problem

Given parameters $\rho^{\text{in}}, \rho^{\text{out}}, a, R, V$ find surface $\mathcal{S}$ and stream function ψ solution to

$$\begin{cases} -\Delta\psi = \dfrac{15}{2} a\, s \sin\theta e_\varphi \mathbb{1}_{\mathcal{D}^{\text{in}}} & \text{in } \mathbb{R}^3 \setminus \mathcal{S} \\ \psi = \dfrac{V}{2} s \sin\theta e_\varphi & \text{on } \mathcal{S} \\ \dfrac{1}{2}\left[\!\left[\rho \left|\operatorname{curl}\psi - V e_3\right|^2\right]\!\right] + \sigma H = \mathrm{const} & \text{on } \mathcal{S} \end{cases}$$

Spherical coordinates $(s, \theta, \varphi) \in [0, \infty) \times [0, \pi) \times [0, 2\pi)$.

The overdetermined free boundary value problem

Given parameters $\rho^{\text{in}}, \rho^{\text{out}}, \sigma, a, R, V$ find a surface $\mathcal{S}$ and a stream function ψ solution to

$$\begin{cases} -\Delta\psi = \dfrac{15}{2} a\, s \sin\theta e_\varphi \mathbb{1}_{\mathcal{D}^{\text{in}}} & \text{in } \mathbb{R}^3 \setminus \mathcal{S} \\ \psi = \dfrac{V}{2} s \sin\theta e_\varphi & \text{on } \mathcal{S} \\ \dfrac{1}{2}\left[\!\left[\rho \left|\operatorname{curl}\psi - V e_3\right|^2\right]\!\right] + \sigma H = \mathrm{const} & \text{on } \mathcal{S} \end{cases}$$

Weber number: $\mathrm{We} = \dfrac{\rho^{\text{out}} V^2 R}{\sigma}$

Vortex Weber number: $\gamma = \dfrac{\rho^{\text{in}} a^2 R^5}{\sigma}$

The overdetermined free boundary value problem

Rescale to $R = 1$ and decompose

$$\psi = \left(a\psi^{\text{in}} + \frac{V}{2} s \sin\theta \, e_\varphi \right) \mathbb{1}_{\mathcal{D}^{\text{in}}} + V\psi^{\text{out}} \mathbb{1}_{\mathcal{D}^{\text{out}}},$$

with $\psi^{\text{in}} \colon \mathcal{D}^{\text{in}} \to \mathbb{R}^3$ solution to

$$\begin{cases} -\Delta\psi^{\text{in}} = \frac{15}{2} s \sin\theta \, e_\varphi & \text{in } \mathcal{D}^{\text{in}}, \\ \psi^{\text{in}} = 0 & \text{on } \mathcal{S}, \end{cases}$$

and $\psi^{\text{out}} \colon \mathcal{D}^{\text{out}} \to \mathbb{R}^3$ vanishing at infinity and solving

$$\begin{cases} -\Delta\psi^{\text{out}} = 0 & \text{in } \mathcal{D}^{\text{out}}, \\ \psi^{\text{out}} = \frac{1}{2} s \sin\theta \, e_\varphi & \text{on } \mathcal{S}. \end{cases}$$

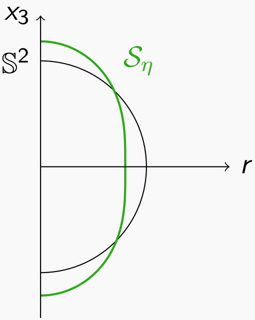
Jump condition: $\dfrac{\gamma}{2} |\operatorname{curl} \psi^{\text{in}}|^2 - \dfrac{\mathrm{We}}{2} |\operatorname{curl} \psi^{\text{out}} - e_3|^2 + H = \text{const}$ on $\mathcal{S}$.

Perturbation of the spherical solution

For a shape function $\eta \in \mathsf{H}^{\beta}(\mathbb{S}^2)$ we consider

$$\mathcal{S}_\eta = \big\{(1+\eta(x))x : x \in \mathbb{S}^2\big\}.$$

In axisymmetric coordinates:



Perturbation of the spherical solution

For a shape function $\eta \in \mathsf{H}^\beta(\mathbb{S}^2)$ we consider

$$\mathcal{S}_\eta = \left\{(1+\eta(x))x : x \in \mathbb{S}^2\right\},$$

with $\mathcal{D}_\eta^{\mathsf{in}}$ and $\mathcal{D}_\eta^{\mathsf{out}}$ well-defined if $\eta > -1$.

We impose

- axi-symmetry $\eta = \eta(\theta)$, and
- reflection invariance across the reference plane, $\eta(\frac{\pi}{2} - \theta) = \eta(\frac{\pi}{2} + \theta)$

and write $\mathsf{H}^\beta_{\mathrm{sym}}(\mathbb{S}^2)$ for that subspace.

Set $\mathcal{M}^\beta = \{\eta \in \mathsf{H}^\beta_{\mathrm{sym}}(\mathbb{S}^2) : \left|\mathcal{D}_\eta^{\mathsf{in}}\right| = \frac{4}{3}\pi$ and $\|\eta\|_{\mathsf{H}^\beta} \leq c_0\}$ for $c_0 > 0$ small.

Perturbative ansatz

We introduce the functional $\mathcal{F}\colon \mathbb{R}\times\mathbb{R}\times\mathcal{M}^{\alpha+2}\to \mathsf{H}^{\alpha}_{\mathrm{sym}}(\mathbb{S}^2)/_{\mathrm{const}}$ as

$$\mathcal{F}(\gamma,\mathrm{We},\eta)=\frac{\gamma}{2}\left|(\operatorname{curl}\psi^{\mathrm{in}}_{\eta})\circ\chi_\eta\right|^2-\frac{\mathrm{We}}{2}\left|(\operatorname{curl}\psi^{\mathrm{out}}_{\eta})\circ\chi_\eta-e_3\right|^2+H_\eta\circ\chi_\eta$$

where $\chi_\eta=(1+\eta(x))x$.

Goal: find We, γ and η such that

$$\mathcal{F}(\gamma,\mathrm{We},\eta)=\mathrm{const}.$$

Spherical solution:

$$\mathcal{F}(\gamma,\gamma,0)=2=\mathrm{const}.$$

$(1)\ \mathcal{F}(\gamma, \mathrm{We}, \eta) = \mathrm{const}$

Theorem (MNS '25):

Let $\beta > 2$. There exists $c_0 = c_0(\beta) > 0$ and an increasing sequence $\Gamma = (\gamma_k)_{k\in\mathbb{N}}$ of positive numbers diverging to infinity with:

(A) For any $\gamma \in [0,\infty) \setminus \Gamma$ and any We close to but different from γ, there exists a unique nontrivial (smooth) solution $\eta = \eta(\gamma, \mathrm{We}) \in \mathcal{M}^\beta$ to the jump equation (1).

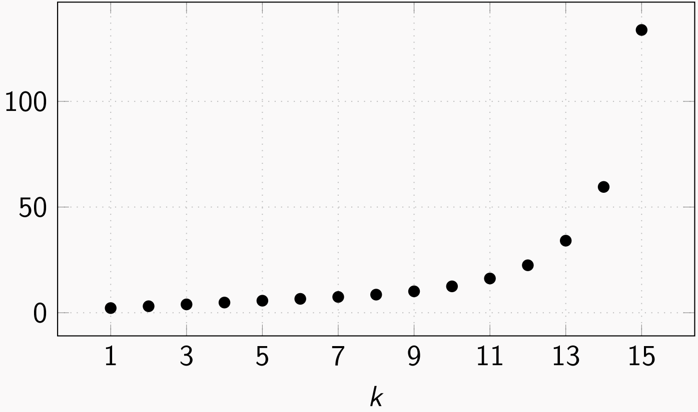
If $\gamma = \varepsilon\delta^{\mathsf{in}}$ and $\mathrm{We} = \varepsilon\delta^{\mathsf{out}}$ for two nonnegative constants $\delta^{\mathsf{in}} \neq \delta^{\mathsf{out}}$ and a small parameter ε, we have the asymptotic expansion

$$\eta_\varepsilon = \varepsilon\frac{3}{32}(\delta^{\mathsf{in}} - \delta^{\mathsf{out}})\left(3\cos^2\theta - 1\right) + o(\varepsilon) \text{ as } \varepsilon \to 0.$$

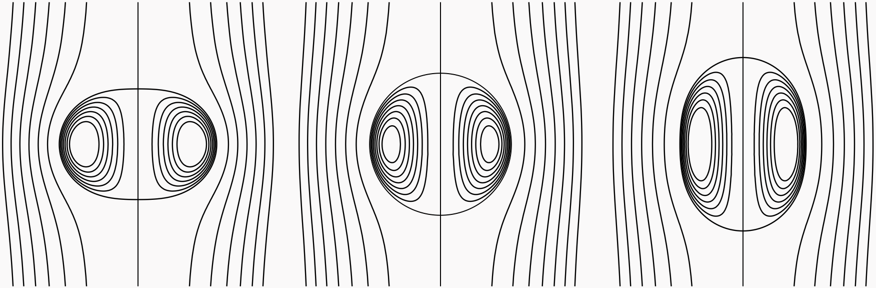
(B) For any $k \in \mathbb{N}$, there exists a unique local curve $s \mapsto \gamma(s)$ passing through γ_k and there are associated nontrivial (smooth) shape functions $\eta(s) \in \mathcal{M}^\beta$ such that the equation (1) is solved at $(\gamma(s), \gamma(s), \eta(s))$.

Remarks on the Theorem

k	1	2	3	4	5	6
γ_k	2.20516	3.07529	3.94492	4.81679	5.69137	6.56836

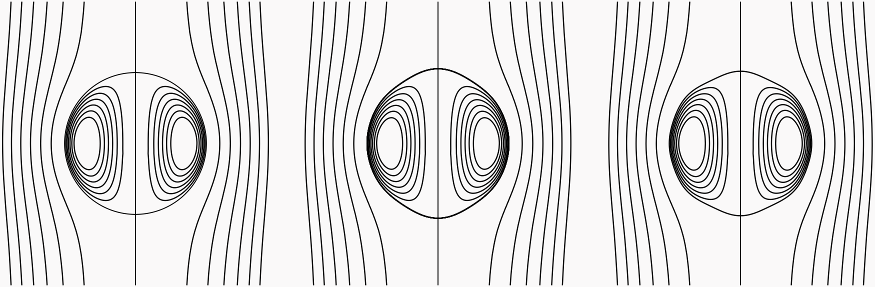


Remarks on the Theorem (A)



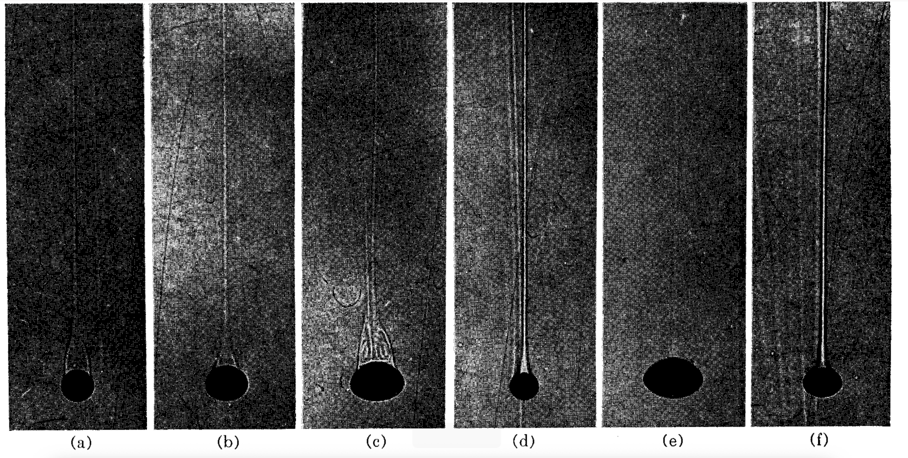
$\mathrm{We} > \gamma$ $\quad$ $\mathrm{We} = \gamma$ $\quad$ $\mathrm{We} < \gamma$

Remarks on the Theorem (B)



$\mathrm{We} = \gamma$ $\quad\quad$ γ_1 $\quad\quad$ γ_2

Remarks on the Theorem



Experimental observations

Droplet Motion in Purified Systems, S. Winnikow and B. T. Chao (1966)

Corollary (MNS '25):

There exist values of γ close to the bifurcation set Γ for which non-spherical steady vortex solutions with $\mathrm{We} = \gamma$ exist. In particular, for these values, the spherical vortex is non-unique.

In the one fluid setting ($\rho^{\mathsf{in}} = \rho^{\mathsf{out}}$) without surface tension ($\sigma = 0$) the spherical solution with Hill's vortex core is unique
up to translations (Amick-Fraenkel '86).

Remarks on the Theorem

Physics literature:

- Moore '58 derives the formal asymptotics of the shape for small Weber numbers neglecting the internal motion ($\rho^{\text{in}} = 0$).
- Harper '72 explains that the inner circulation can be approximated by Hill's vortex core.
- Pozrikidis '89 provides numerical evidence of the bifurcation branch and finds approximations for γ_1, γ_2.

Remarks on the Theorem

Mathematical literature:

- Crowdy-Wegmann '00 investigate two-dimensional vortex sheets
- Meyer-Seis '24 construct bubble rings
- Baldi-La Manna-La Scala '25 construct rotating solutions of close-to-spherical shape
- Murgante-Roulley-Scrobogna '25 investigate the dynamics of of close-to-spherical vortex sheets

$(1)\ \mathcal{F}(\gamma, \mathrm{We}, \eta) = \mathrm{const}$

Theorem (MNS '25):

Let $\beta > 2$. There exists $c_0 = c_0(\beta) > 0$ and an increasing sequence $\Gamma = (\gamma_k)_{k\in\mathbb{N}}$ of positive numbers diverging to infinity with:

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(B) For any $k \in \mathbb{N}$, there exists a unique local curve $s \mapsto \gamma(s)$ passing through γ_k and there are associated nontrivial (smooth) shape functions $\eta(s) \in \mathcal{M}^\beta$ such that the equation (1) is solved at $(\gamma(s), \gamma(s), \eta(s))$.

Idea of the proof

Recall

$$\mathcal{M}^{\alpha+2} = \left\{ \eta \in \mathsf{H}^{\alpha+2}_{\mathrm{sym}}(\mathbb{S}^2) : \left|\mathcal{D}^{\mathsf{in}}_\eta\right| = \frac{4}{3}\pi \text{ and } \|\eta\|_{\mathsf{H}^{\alpha+2}} \leq c_0 \right\}$$

c_0 small; $\alpha > 0$ ($\rightsquigarrow$ Sobolev embedding; algebra property),

$$\mathcal{F} \colon \mathbb{R} \times \mathbb{R} \times \mathcal{M}^{\alpha+2} \to \mathsf{H}^{\alpha}_{\mathrm{sym}}(\mathbb{S}^2)/_{\mathrm{const}},$$

$$\mathcal{F}(\gamma, \mathrm{We}, \eta) = \frac{\gamma}{2}\left|(\mathrm{curl}\,\psi^{\mathsf{in}}_\eta) \circ \chi_\eta\right|^2 - \frac{\mathrm{We}}{2}\left|(\mathrm{curl}\,\psi^{\mathsf{out}}_\eta) \circ \chi_\eta - e_3\right|^2 + H_\eta \circ \chi_\eta$$

$$\mathcal{F}(\gamma, \gamma, 0) = \mathrm{const} \text{ for all } \gamma > 0.$$

Idea of the proof

Observe that $\mathcal{F}$ is Fréchet differentiable.

We calculate

$$\mathrm{D}_\eta \mathcal{F}(\gamma, \gamma, \eta)\big|_{\eta=0} : T_0\mathcal{M}^{\alpha+2} \to \mathsf{H}^{\alpha}_{\mathrm{sym}}(\mathbb{S}^2)/_{\mathrm{const}},$$

which turns out to be invertible precisely if $\gamma \notin \Gamma = (\gamma_k)_{k\in\mathbb{N}}$.

- If $\gamma \notin \Gamma$ we can employ the implicit function theorem to deduce part (A).

- At $\gamma \in \Gamma$ we perform a bifurcation analysis by employing the Crandall-Rabinowitz theorem to prove part (B).

The curvature term

The curvature of a graph over the sphere can be written as

$$H_\eta \circ \chi_\eta = \frac{1}{1+\eta}\left(2\frac{1+\eta}{\sqrt{g_\eta}} - \frac{\Delta_{\mathbb{S}^2}\eta}{\sqrt{g_\eta}} - \nabla_{\mathbb{S}^2}\frac{1}{\sqrt{g_\eta}} \cdot \nabla_{\mathbb{S}^2}\eta\right)$$

where $g_\eta = (1+\eta)^2 + |\nabla_{\mathbb{S}^2}\eta|^2$.

We deduce

$$\mathrm{D}_\eta H_\eta \circ \chi_\eta|_{\eta=0} = -(\Delta_{\mathbb{S}^2} + 2\mathrm{Id}).$$

Recall that the spherical harmonics $\{Y_l^m : l \in \mathbb{N}, -l \leq m \leq l\}$ form an eigenbasis for this operator with eigenvalues $(l+2)(l-1)$.

A first part of the proof

At $\gamma = 0$ the curvature term dominates

$$\mathrm{D}_\eta \mathcal{F}(0,0,\eta)\big|_{\eta=0} = -\left(\Delta_{\mathbb{S}^2} + 2\mathrm{Id}\right) : \, T_0\mathcal{M}^{\alpha+2} \to \mathsf{H}^\alpha_{\mathrm{sym}}(\mathbb{S}^2)/_{\mathrm{const}}.$$

Moreover, $T_0\mathcal{M}^{\alpha+2} = \left\{\eta \in \mathsf{H}^{\alpha+2}_{\mathrm{sym}}(\mathbb{S}^2) : \int_{\mathbb{S}^2} \eta \mathrm{d}\sigma = 0\right\}$ and

$$\mathsf{H}^\beta_{\mathrm{sym}}(\mathbb{S}^2) := \left\{f \in \mathsf{H}^\beta(\mathbb{S}^2) : \langle f, Y_l^m \rangle = 0 \text{ if } l \text{ is odd or } m \neq 0\right\}.$$

Hence, the operator is invertible and we can locally solve

$$\mathcal{F}\left(\varepsilon\delta^{\mathrm{in}}, \varepsilon\delta^{\mathrm{out}}, \eta_\varepsilon\right) = \mathrm{const}, \; \varepsilon \in [0, \varepsilon_0)$$

for functions $(\eta_\varepsilon)_{\varepsilon\in[0,\varepsilon_0)}$. Noting that

$$\mathrm{D}_\varepsilon \mathcal{F}\left(\varepsilon\delta^{\mathrm{in}}, \varepsilon\delta^{\mathrm{out}}, 0\right)\big|_{\varepsilon=0} = -\frac{3}{2}\sqrt{\frac{\pi}{5}}(\delta^{\mathrm{in}} - \delta^{\mathrm{out}})Y_2^0(\theta)$$

gives the first-order asymptotics.

The jump term

A longer calculation reveals that

$$\langle \mathrm{D}_\eta \mathcal{F}(\gamma,\gamma,\eta)|_{\eta=0}, \delta\eta\rangle = \frac{9}{2}\gamma \sin\theta\, e_\varphi \cdot (2\mathrm{Id} - \Lambda)(\sin\theta\,\delta\eta\, e_\varphi) - (\Delta_{\mathbb{S}^2} + 2\mathrm{Id})\,\delta\eta,$$

where Λ is the Dirichlet-to-Neumann map for the Laplacian on the unit ball in $\mathbb{R}^3$.

Analysis of the linearisation

We write

$$[\mathcal{A}(\mu)](\delta\eta) = \frac{2}{9\gamma}\langle \mathrm{D}_\eta \mathcal{F}(\gamma,\gamma,\eta)|_{\eta=0}\,, \delta\eta\rangle$$
$$= \sin\theta\, e_\varphi\cdot(2\mathrm{Id}-\Lambda)(\sin\theta\,\delta\eta\, e_\varphi) - \mu(\Delta_{\mathbb{S}^2}+2\mathrm{Id})\delta\eta,$$

for $\mu = \frac{2}{9\gamma}$.

Finding $\mu > 0$ such that $\ker \mathcal{A}(\mu) \neq \{0\}$ is equivalent to the eigenvalue problem of the symmetric and compact operator

$$\mathcal{K} = (\Delta_{\mathbb{S}^2}+2\mathrm{Id})^{-\frac{1}{2}}\sin\theta\, e_\varphi\cdot(2\mathrm{Id}-\Lambda)(\sin\theta\,\left((\Delta_{\mathbb{S}^2}+2\mathrm{Id})^{-\frac{1}{2}}\delta\eta\right)\, e_\varphi)$$

Analysis of the linearisation

In representation via spherical harmonics

$$\delta\eta = \sum_{k=1}^{\infty} v_k Y_{2k}^0(\theta)$$

this is an infinite matrix operator in weighted sequence spaces

$$\mathrm{h}^\alpha := \left\{ v = (v_k)_{k\in\mathbb{N}} \,:\, \|v\|_{\mathrm{h}^\alpha}^2 := \sum_{k=1}^{\infty} k^{2\alpha} v_k^2 < \infty \right\}.$$

Analysis of the linearisation

The operator $\mathcal{K}$ can be written as an infinite Jacobi matrix

$$\mathcal{K} = \begin{pmatrix} A_1 & B_1 & 0 & & & \\ B_1 & A_2 & B_2 & \ddots & & \\ 0 & B_2 & A_3 & B_3 & \ddots & \\ & \ddots & B_3 & A_4 & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & & & \end{pmatrix}$$

$$A_k = -\frac{16k^3 + 4k^2 - 8k - 1}{64k^4 + 112k^3 + 44k^2 - 7k - 3} \sim -\frac{1}{4k}$$

$$B_k = \frac{(k+1)(2k-1)(2k+1)}{(4k+3)\sqrt{64k^6 + 288k^5 + 420k^4 + 180k^3 - 69k^2 - 63k - 10}} \sim \frac{1}{8k}$$

Analysis of the linearisation

Recall: $\mu = \dfrac{2}{9\gamma}$

Lemma:

Let $\alpha \geq 0$.

a) For any $\mu \neq 0$, the operator $\mathcal{A}(\mu) : \mathrm{h}^{\alpha+2} \to \mathrm{h}^{\alpha}$ is a symmetric Fredholm operator of index 0.

b) For $\mu > 0$, the nullspace $N(\mathcal{A}(\mu))$ of $\mathcal{A}(\mu)$ is at most one-dimensional and $N(\mathcal{A}(\mu)) \subset \mathrm{h}^{\beta}$ for all $\beta \geq 0$. Moreover, $N(\mathcal{A}(\mu)) = \{0\}$ for $\mu \leq 0$.

c) There exists a strictly decreasing sequence $(\mu_k)_{k\in\mathbb{N}} \subset \mathbb{R}^+$ with limit 0 such that $\mathcal{A}(\mu_k)$ has a 1-dimensional nullspace and $A(\mu)$ is invertible if $\mu \notin \{\mu_k : k \in \mathbb{N}\} \cup \{0\}$.

d) We have $\mu_1 \leq \frac{\sqrt{2}}{21\sqrt{5}} + \frac{\sqrt{5}}{22\sqrt{13}} + \frac{127}{2079} \approx 0.119394$.

e) For $0 \neq v^k \in N(\mathcal{A}(\mu_k))$, we have the transversality condition

$$\mathrm{D}_\mu \mathcal{A}(\mu)\big|_{\mu=\mu_k} v^k \notin R(\mathcal{A}(\mu_k)).$$

Proof of the Theorem (A)

As before, we employ the implicit function theorem to

$$(\varepsilon, \eta) \mapsto \mathcal{F}\left(\gamma + \varepsilon\delta^{\mathrm{in}}, \gamma + \varepsilon\delta^{\mathrm{out}}, \eta\right)$$

whenever $\gamma \notin \Gamma$ and obtain $(\eta_\varepsilon)_{|\varepsilon|<\varepsilon_0}$ such that

$$\mathcal{F}\left(\gamma + \varepsilon\delta^{\mathrm{in}}, \gamma + \varepsilon\delta^{\mathrm{out}}, \eta_\varepsilon\right) = \mathrm{const} \text{ for all } \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

Proof of the Theorem (B)

Theorem of Crandall and Rabinowitz '71:

Let M be a smooth Banach manifold and Y be a Banach space, $I \subset \mathbb{R}$ some open interval, and $\mathcal{G}\colon I \times M \to Y$ be continuous. Let $w_0 \in M$. If

(1) $\mathcal{G}(\lambda, w_0) = 0$ for all $\lambda \in I$.

(2) The Fréchet derivatives $\mathrm{D}_\lambda \mathcal{G}$, $\mathrm{D}_w \mathcal{G}$, $\mathrm{D}^2_{\lambda w} \mathcal{G}$ exist and are continuous.

(3) There exists $\lambda^* \in I$ and $w^* \in T_{w_0} M$ such that $Y / R(\mathrm{D}_w \mathcal{G}(\lambda^*, w_0))$ and $N(\mathrm{D}_w \mathcal{G}(\lambda^*, w_0)) = \mathrm{span}\,(w^*)$ is 1-dimensional.

(4) $\mathrm{D}^2_{\lambda w} \mathcal{G}(\lambda, w)|_{(\lambda,w)=(\lambda^*,w_0)} w^* \notin R(\mathrm{D}_w \mathcal{G}(\lambda^*, w)|_{w=w_0})$.

Then there exists a continuous local bifurcation curve $\{(\lambda(s), w(s))\}_{|s|<\varepsilon}$ with ε small such that $(\lambda(0), w(0)) = (\lambda^*, w_0)$ and

$$\{(\lambda, w) \in U : w \neq w_0, \mathcal{G}(\lambda, w) = 0\} = \{(\lambda(s), w(s)) : 0 < |s| < \varepsilon\}$$

for some neighbourhood U of $(\lambda^*, w_0) \in I \times M$. Moreover,

$$w(s) = w_0 + s w^* + o(s) \quad \text{in } M,\ |s| < \varepsilon.$$

Proof of the Theorem (B)

We apply the theorem of Crandall-Rabinowitz to

$$\mathcal{F}\colon (0,\infty)\times\mathcal{M}^{\alpha+2}\to \mathsf{H}^{\alpha}_{\mathrm{sym}}(\mathbb{S}^2)/_{\mathrm{const}},\quad (\gamma,\eta)\mapsto \mathcal{F}(\gamma,\gamma,\eta).$$

As

$$\mathrm{D}_{\eta}\mathcal{F}(\gamma,\eta)|_{\eta=0}=\frac{9}{2}\gamma\mathcal{A}\left(\frac{2}{9\gamma}\right)\colon T_0\mathcal{M}^{\alpha+2}\to \mathsf{H}^{\alpha}_{\mathrm{sym}}(\mathbb{S}^2)/_{\mathrm{const}}$$

the assumptions (1)–(4) in the theorem of Crandall and Rabinowitz are a consequence of the previous lemma.

References

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