Kinetic maximal L^p-regularity

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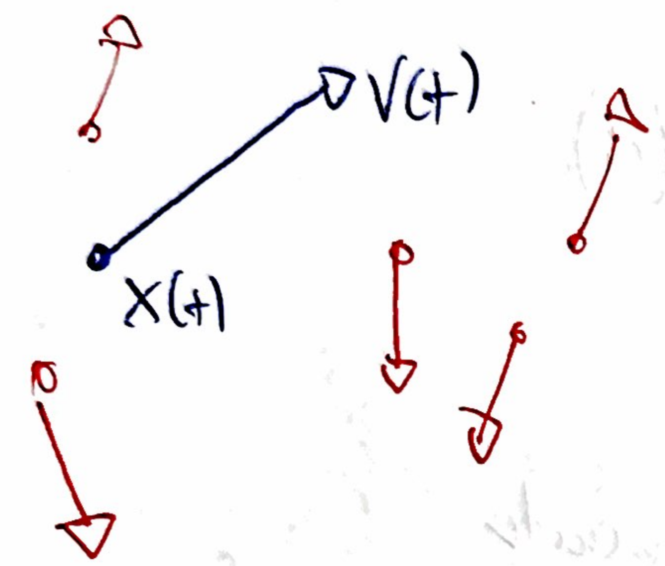
More Particles



$$\begin{cases} V(t) = V_0 \\ X(t) = tV_0 + X_0 \end{cases}$$

Free transport

Particle distribution function

$$u = u(t,x,v) : [0,T] \times \boxed{\mathbb{R}^n} \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

time, position, velocity

other choices are possible $\mathbb{T}^n$, Ω w. bdry cond.

Free transport eq.

$$\begin{cases} \underbrace{\boxed{\partial_t u + v\cdot\nabla_x u}}_{\text{kinetic term}} = f = f(t,x,v) & \text{source term} \\ u(0) = g = g(x,v) & \text{initial value} \end{cases}$$

We want some Interaction

non-collisional
- Vlasov-Poisson eq & others
- charged particles

$\boxed{\text{collisional}}$

Collisional kinetic eq

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = Au + f \\ u(0) = g \end{cases} \quad (1)$$

- A acts in v, possibly non-linear
- $A = A(u,u)$ Boltzmann collision operator
  bilinear, non-local, roughly like $-(-\Delta_v)^{\frac{s}{2}}$

- Landau equation (Plasma physics)

$$A = A(u)u = \bar{a}(u) : \nabla_v^2 u + \bar{c}(u) u$$

$\gamma \in [-n, 0)$

$$\bar{a}(u) = a_{\gamma n} \int_{\mathbb{R}^n} \left(Id - \frac{w}{|w|} \otimes \frac{w}{|w|} \right) |w|^{\gamma+2} u(t,x,v-w)\, dw$$

$$\approx (1+|v|)^{\gamma+2} \int u(t,x,w)\, dw \quad \text{(roughly)}$$

$$\bar{c}(u) = \begin{cases} c_{\gamma,n} \int |w|^{\gamma} u(t,x,v-w)\, dw & \gamma \in (-n,0) \\ c_n u & \gamma = -n \end{cases}$$

Linear first – Kolmogorov equation (1934)

$$\begin{cases} \partial_t u + \underbrace{\boxed{v \cdot \nabla_x u}}_{\text{1. Unbounded}} = \underbrace{\boxed{\Delta_v} u}_{\text{2. Degenerate}} + f \\ u(0) = g \end{cases} \qquad (1)$$

Comments:

1., 2., 3. hypoelliptic, 4. scaling invariance

$$\cancel{(t,x,v)} \overset{\lambda}{\mapsto} (\lambda^2 t, \lambda^3 x, \lambda v)$$

translation invariance

$$(t_0, x_0, v_0) \mapsto (t - t_0,\ x - x_0 - (t - t_0)v,\ v - v_0)$$

Goal: maximal/optimal solution theory

Find spaces $\begin{cases} u \in Z \\ f \in X \\ g \in X_\gamma \end{cases}$ such that

$\exists!$ sol. $u \in Z$ to $(1)/(2)$ $\Leftrightarrow$ $f \in X$, $g \in X_\gamma$

$\Leftrightarrow$: kinetic maximal L^p-regularity

$\Rightarrow X = L^p((0,T); L^p(\mathbb{R}^{2n})) = L^p L^p$, $p \in (1, \infty)$

strong L^p-solutions

Solution space for strong L^p-solutions

$Z = \{u : u, \partial_t u, \Delta_v - v\cdot\nabla_x u \in L^p L^p\}$?

No! $\sigma_p(\Delta_v - v\cdot\nabla_x) = \{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$

$\Rightarrow$ no maximal L^p-regularity

Better:

$Z = \{u : u, \partial_t u + v\cdot\nabla_x u, \Delta_v u \in L^p L^p\}$.

Definition
Let $p \in (1,\infty)$.
We say that a linear operator $A : D(A) \subseteq L^p(\mathbb{R}^{2d}) \to L^p(\mathbb{R}^{2d})$ admits kin. max L^p-regularity if $\forall f \in L^p L^p$
$\exists!$ sol $u \in Z = \{u : u, \partial_t u + v\cdot\nabla_x u, Au \in L^p L^p\}$

of (1) or $\begin{cases} \partial_t u + v\cdot\nabla_x u = Au + f \\ u(0) = g \end{cases}$.
w. $g=0$

Questions:

1. Does it work for $A = \Delta_v$?

2. What about X_y ?

To 1: Fundamental solution:

Fourier $(x,v) \mapsto (k,\xi)$

$$\begin{cases} \partial_t \hat{u} - k \cdot \nabla_\xi \hat{u} = -|\xi|^2 \hat{u} \\ \hat{u}(0) = \hat{g} \end{cases}$$

$$\hat{w}(t,k,\xi) = \hat{u}(t,k,\xi - tk)$$

$$\rightsquigarrow \begin{cases} \partial_t \hat{w} = -|\xi - tk|^2 \hat{w} \\ \hat{w}(0) = \hat{g} \end{cases}$$

$$\Rightarrow \hat{w}(t,k,\xi) = \exp\left(-\int_0^t |\xi - sk|^2 ds\right) \hat{g}(k,\xi)$$

$$\Rightarrow \hat{u}(t,k,\xi) = \exp\left(-\int_0^t |\xi + (t-s)k|^2 ds\right) \hat{g}(k,\xi + tk)$$

$\sim -t|\xi|^2 - t^3|k|^2$

in physical variable

$$G_2(t,x,v) = \frac{c}{t^{2n}} \exp\left(-\frac{|v|^2}{t} + \frac{3}{t^2}\langle v,x\rangle - \frac{3}{t^3}|x|^2\right)$$

Given smooth f, g

$$u(t,x,v) = \int_{\mathbb{R}^{2n}} G_2(t, x-y-tw, v-w)\, g(y,w)\, d(y,w) \qquad (3)$$
$$+ \int_0^t \int_{\mathbb{R}^{2n}} G_2(t-s, x-y-(t-s)w, v-w)\, f(s,y,w)\, d(y,w)\, ds$$

- $f=0,\ g\in L^p \Rightarrow u \in C^\infty((0,\infty)\times\mathbb{R}^{2n})$

  ?! smooth in x ?!

- not really of convolution type.

- considering $\Delta_v u$ leads to a singular integral.
  $(g=0)$

Apply singular integral theory on homogeneous groups developed by Folland & Stein '74

to obtain

Theorem

Let $p \in (1,\infty)$. Δ_v admits kin. max. L^p-reg.

$\forall f \in L^p$: u as in (2) solves (1) &

$$\|\partial_t u + v\cdot\nabla_x u\|_p + \|\Delta_v u\|_p \leq C \|f\|_p$$

to 2. X_γ?

- temporal trace is well-defined

$$\{u : u, \partial_t u + v\cdot\nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$
$$\hookrightarrow \mathcal{C}([0,T]; L^p(\mathbb{R}^{2n})\}$$

- trace space

$$X_\gamma = \{g : \exists u \in Z : u(0) = g\} \qquad \|g\|_{X_\gamma} = \inf_{\substack{u \in Z \\ u(0)=g}} \|u\|_Z$$

& $Z \hookrightarrow \mathcal{C}([0,T]; X_\gamma)$

- A KMR $\Rightarrow$ $\exists!$ u ~~sol.~~ $\in Z$ sol to (1)

$\Leftrightarrow$ $g \in X_\gamma$, $f \in X$

kinetic regularization (Bouchut '02)

Thm

$u \in L^p(\mathbb{R}^{1+2n})$ with $\partial_t u + v \cdot \nabla_x u = f \in L^p(\mathbb{R}^{1+2n})$

& $\Delta_v u \in L^p(\mathbb{R}^{1+2n})$

$\Rightarrow (-\Delta_x)^{\frac{1}{3}} = D_x^{\frac{2}{3}} u \in L^p(\mathbb{R}^{1+2n})$

$\hat{=} \quad |k|^{\frac{2}{3}} \hat{u}$

Proof ($p=2$) by Alexandre '12

Fourier $(x,v) \mapsto (k,\xi)$

$\partial_t \hat{u} - k \cdot \nabla_\xi \hat{u} = \hat{f}$ $| \cdot \bar{\hat{u}}$ & Cauchy-Schwarz

$\partial_t |\hat{u}|^2 - k \cdot \nabla_\xi |\hat{u}|^2 \lesssim |\hat{f}|\,|\hat{u}|$

Method of characteristic:

$$|\hat{u}|^2(t,k,\xi) \lesssim \int_{-\infty}^{t} |\hat{f}\hat{u}|\,(t-s,k,\xi+sk)\,ds$$

Goal: $|k|^{\frac{2}{3}} \hat{u} \in L^2$

Fix $h \in \mathbb{R}^n$

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |k|^{4/3} |\hat{u}(t,h,\xi)|^2 \, d\xi \, dt$$

$$= \int_{\mathbb{R}}\int_{|\xi| \geq |k|^{\frac{1}{3}}} \dots \quad + \underbrace{\int_{\mathbb{R}}\int_{|\xi| \leq |k|^{\frac{1}{3}}} \dots}_{B}$$

good part

$|k|^{4/3} \leq |\xi|^4$ ✓

th

$$B \leq \int_{\mathbb{R}}\int_{|\xi| \leq |k|^{\frac{1}{3}}} |k|^{4/3} \int_{-\infty}^{t} |\hat{f}\hat{u}|(t-s, h, \xi+sh) \, ds \, d\xi \, dt$$

$$= \int_{|\xi| \leq |k|^{\frac{1}{3}}} |k|^{\frac{4}{3}} \int_{\mathbb{R}} \int_{0}^{\cancel{\infty}} |\hat{f}\hat{u}|(\underbrace{t-s}_{=\tilde{t}}, h, \xi+sh) \, d\tilde{t} \, ds \, d\xi$$

$$\leq \int_{\mathbb{R}}\int_{\mathbb{R}} \int_{|\tilde{\xi}-sh| \leq |k|^{\frac{1}{3}}} |k|^{\frac{4}{3}} |\hat{f}\hat{u}|(\tilde{t}, h, \tilde{\xi}) \, d\tilde{\xi} \, ds \, d\tilde{t} \qquad \cancel{dt}$$

$$\big||\tilde{\xi}| - s|k|\big| \leq |\tilde{\xi} - sh| \leq |k|^{\frac{1}{3}} \text{ AND } \left|\frac{|\tilde{\xi}|}{|k|} - s\right| \leq |k|^{-\frac{2}{3}}$$

$$\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}^d} |k|^{\frac{2}{3}} |\hat{u}|^2 |\hat{f}|^2 \, d\xi \, dt$$

Young

$\int dk$ & absorb

Theorem (kinetic trace) (for kol. eq.)

$p \in (1,\infty)$

$$X_\gamma \cong B^{\frac{2}{3}(1-\frac{1}{p})}_{pp,x}(\mathbb{R}^{2n}) \cap B^{2(1-\frac{1}{p})}_{pp,v}(\mathbb{R}^{2n})$$

Proof:

- $\partial_t + v \cdot \nabla_x$ is not a classical trace/interpol. operator
- by hand with fund sol., characteristic & kin regularisation

suitable Def of anisotropic Besov-space