# Trajectories and the De Giorgi-Nash-Moser theory

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# Kinetic equations

Here: $(t, x, v) \in \Omega_T = (0, T) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2n}$.

Study particle density $f = f(t, x, v) \colon \Omega_T \to \mathbb{R}$

# Kolmogorov equation

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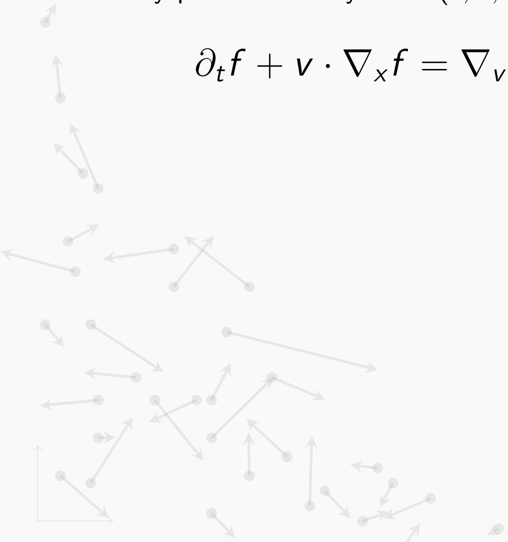
$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\mathfrak{a}(t, x, v) \nabla_v f)$$



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with $\mathfrak{a}\colon \Omega_T \to \mathbb{R}^{n \times n}$ measurable such that

(H1) $\lambda |\xi|^2 \leq \langle \mathfrak{a}(t, x, v)\xi, \xi \rangle$ for all $\xi \in \mathbb{R}^n$ and a.e. $(t, x, v) \in \Omega_T$

(H2) $\sum_{i,j=1}^{n} |\mathfrak{a}_{ij}(t, x, v)|^2 \leq \Lambda^2$ for a.e. $(t, x, v) \in \Omega_T$

and some constants $0 < \lambda < \Lambda$.

# Kinetic geometry

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f$$

Scaling invariance:

$\lambda \mapsto (\lambda^2 t, \lambda^3 x, \lambda v)$

Translation invariance:

$(t_0, x_0, v_0) \mapsto (t - t_0, x - x_0 - (t - t_0) v_0, v - v_0)$

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Kinetic cylinders:

$Q_r(t_0, x_0, v_0)$

$= \{-r^2 < t - t_0 \leq 0,\ |x - x_0 - (t - t_0)v_0| < r^3,\ |v - v_0| < r^3\}$

Can work at unit scale from now on.

# Energy estimate



Testing (1) with $f\varphi^2$ for a cutoff function $\varphi$ yields (formally):

$$\sup_{t\in(-1,0]} \int_{B_1(0)} |f(t,\cdot)|^2 \, \mathrm{d}(x,v) + \int_{-1}^0 \int_{B_1(0)} |\nabla_v f|^2 \, \mathrm{d}(t,x,v) \lesssim \int_{-2}^0 \int_{B_2(0)} |f|^2 \, \mathrm{d}(t,x,v)$$

Natural solution space

$$L^\infty_t(L^2_{x,v}) \cap L^2_{t,x}(\dot{H}^1_v)$$

# Weak solutions



Definition:

A function $f \in L^\infty_t L^2_{x,v}(\Omega_T) \cap L^2_{t,x} \dot{H}^1_v(\Omega_T)$ is a weak (sub-, super-) solution to (1) if for all $\varphi \in C_c^\infty(\Omega_T)$ with $\varphi \geq 0$ we have

$$\int_{(0,T)\times\mathbb{R}^{2n}} \Big[ - f(\partial_t + v \cdot \nabla_x)\varphi + \langle \mathfrak{a} \nabla_v f, \nabla_v \varphi \rangle \Big] \mathrm{d}(t,x,v) = (\geq, \leq)\, 0.$$

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Literature:

- Regularity, existence and uniqueness of weak solutions together with P. Auscher and C. Imbert 24
- previous works: Carrillo 98, Albritton-Armstrong-Mourrat-Novack 24, N.-Zacher 21, Nyström-Litsgård 21

What can we say *a priori* about the regularity of weak (sub-, super-) solutions?

# Kinetic De Giorgi-Nash-Moser theory

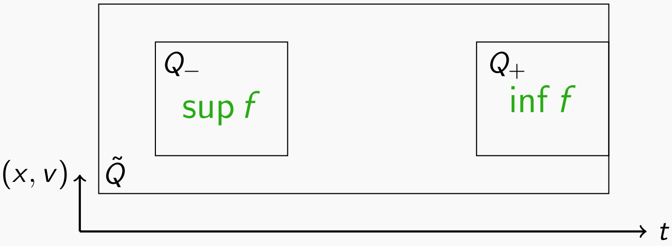
# Harnack inequality

$(1)\ \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\mathfrak{a} \nabla_v f)$

Theorem (GIMV 19, GI 22, GM 22):

There exists a universal const $C = C(n, \lambda, \Lambda) > 0$ such that for any nonnegative weak solution $f$ of (1) in $\tilde{Q}$ we have

$$\sup_{Q_-} f \leq C \inf_{Q_+} f.$$



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# Kinetic De Giorgi-Nash-Moser theory

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# De Giorgi-Nash-Moser theory

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XXIV, 727–740 (1971)

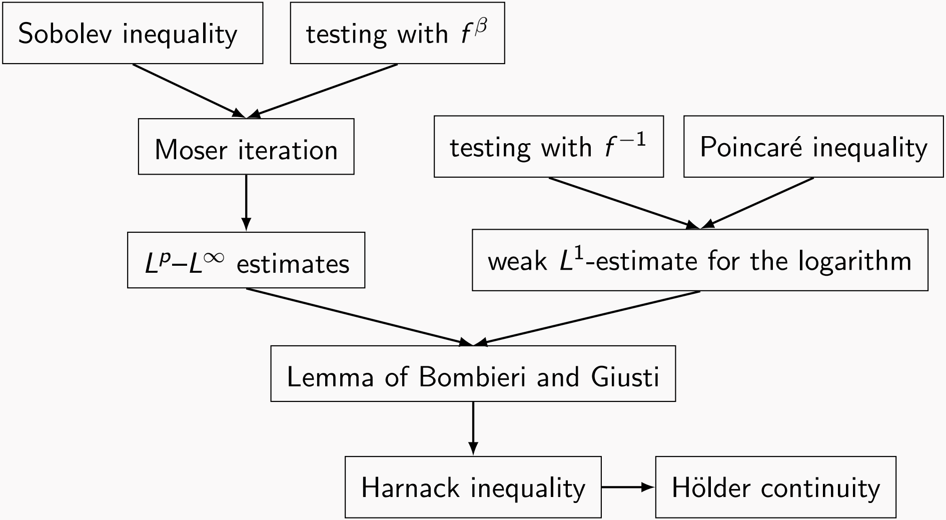
## On a Pointwise Estimate for Parabolic Differential Equations*

J. MOSER

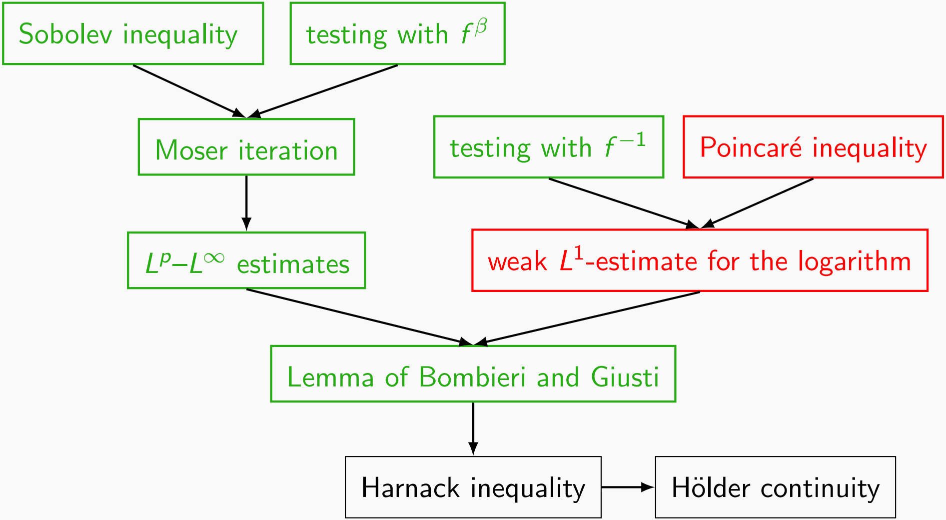
**§1.** The purpose of this note is to describe a simplified proof of a theorem on linear parabolic differential equations which was published earlier in this journal (cf. [6]). This theorem gives a pointwise estimate for positive weak solutions of linear parabolic differential equations and is usually referred to as the Harnack inequality since it generalizes a classical inequality by Harnack for positive harmonic functions. The proof of this theorem for parabolic equations with variable coefficients uses a collection of *a priori* estimates for the powers and the logarithm of the solutions which are played out against each other with the help of general inequalities, primarily consequences of Sobolov's inequality. At one point, however, our previous argument required a new estimate (called Main Lemma in [6]) which generalizes an interesting theorem by F. John and L. Nirenberg. The proof of this lemma is quite intricate and it was desirable to avoid it entirely.

# Moser's 1971 method in kinetic theory

# Moser's 1971 method in parabolic theory



# Towards Moser's 1971 method in kinetic theory



# The logarithm

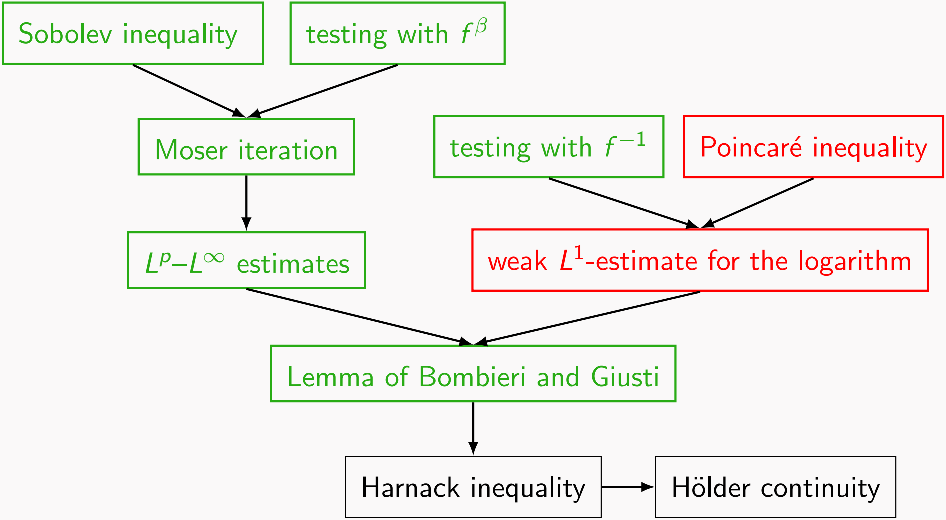
Suppose that $f$ is a positive weak supersolution to

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\mathfrak{a}(t,x,v)\nabla_v f)$$

then the $g = \log f$ is a weak supersolution to

$$\partial_t g + v \cdot \nabla_x g = \nabla_v \cdot (\mathfrak{a}\nabla_v g) + \langle \mathfrak{a}\nabla_v g, \nabla_v g\rangle.$$

# Towards Moser's 1971 method in kinetic theory



# Jerison's Poincaré inequality

Theorem (Jerison 86):

Let $X_0, \ldots, X_m$ be smooth vector fields satisfying Hörmanders rank condition. Then,

$$\int_{B_r} |f - f_{B_r}|^2 \, \mathrm{d}x \leqslant C r^2 \int_{B_r} \sum_{i=0}^{m} |X_i f|^2 \, \mathrm{d}x.$$

Here, $B_r$ are balls with respect to a natural metric.

# Jerison's Poincaré inequality - kinetic?

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Kinetic: $X_0 = \partial_t + v \cdot \nabla_x$ and $X_i = \partial_{v_i}$, $i = 1, \ldots, n$

# Jerison's Poincaré inequality - kinetic?

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We have

$$\int_{Q_r} |f - f_{Q_r}|^2 \,\mathrm{d}(t,x,v) \leqslant Cr^2 \int_{Q_r} |\partial_t f + v \cdot \nabla_x f|^2 + |\nabla_v f|^2 \,\mathrm{d}(t,x,v).$$

Here, $Q_r$ are kinetic cylinders.

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Here, $Q_r$ are kinetic cylinders.

Need to treat $\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot h$, for some $h \in L^2$ at the correct scale.

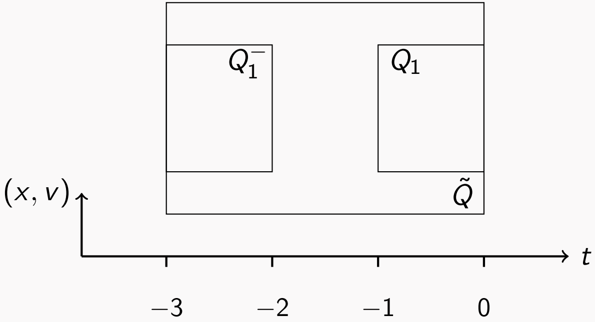
# Kinetic Poincaré inequality



Theorem (Guerand & Mouhot 22, N. & Zacher 22):

Let $h \in L^1(\tilde{Q}; \mathbb{R}^n)$ and $\varphi^2$ be supported in $Q_1^-$. Then, there exists a constant $C = C(n, \varphi) > 0$ such that for all subsolutions $f \geq 0$ to (1) in $\tilde{Q}$ we have

$$\left\| (f - \langle f\varphi^2 \rangle_{Q_1^-})_+ \right\|_{L^1(Q_1)} \leq C \left( \|\nabla_v f\|_{L^1(\tilde{Q})} + \|h\|_{L^1(\tilde{Q})} \right)$$



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Spacetime Poincaré inequalities are "too weak".

# Trajectories

Euclidean $f(v)$ - Poincaré inequality:

$$f(v) - f(w) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} f\left(w + r(v-w)\right) \mathrm{d}r$$

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Parabolic $f(t,v)$

$$f(t,v) - f(\eta,w) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} f(\gamma(r)) \mathrm{d}r$$

with $\gamma\colon [0,1] \to \mathbb{R} \times \mathbb{R}^n$ with $\gamma(0) = (\eta, w)$ and $\gamma(1) = (t, v)$.

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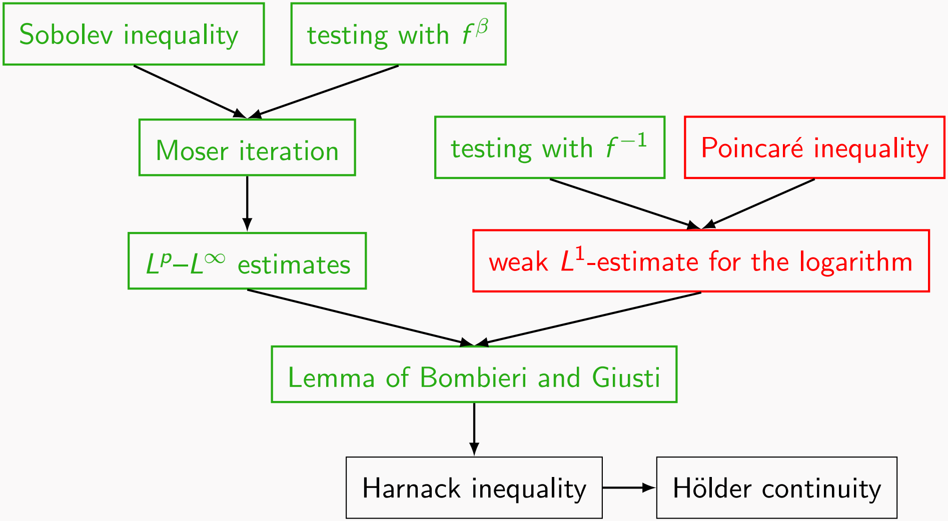
with $\gamma \colon [0, 1] \to \mathbb{R} \times \mathbb{R}^n$ with $\gamma(0) = (\eta, w)$ and $\gamma(1) = (t, v)$.

Parabolic trajectory: $\gamma(r) = (\eta + r(t - \eta), w + r^{1/2}(v - w))$

# Towards Moser's 1971 method in kinetic theory



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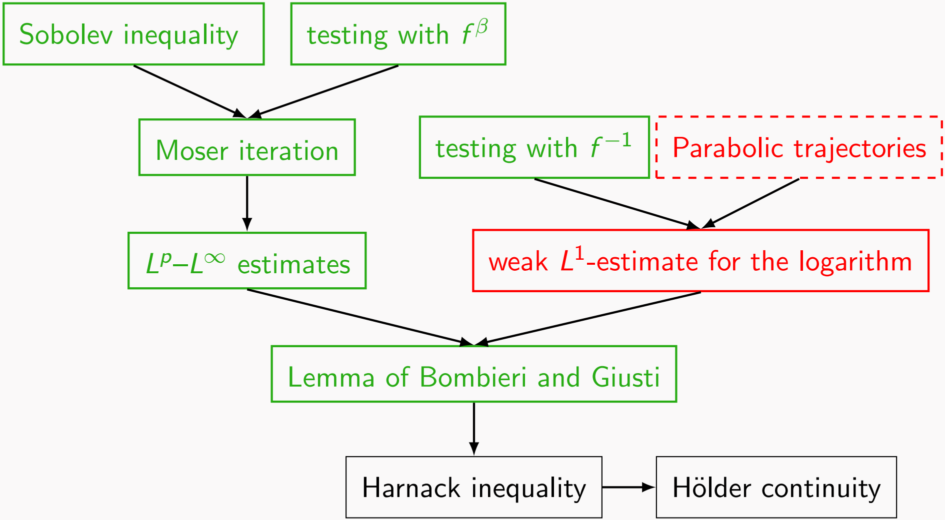

# Kinetic trajectories

Can we walk from $(t, x, v)$ to $(\eta, y, w)$ along $\partial_t + v \cdot \nabla_x$ and $\partial_{v_1}, \dots \partial_{v_n}$?

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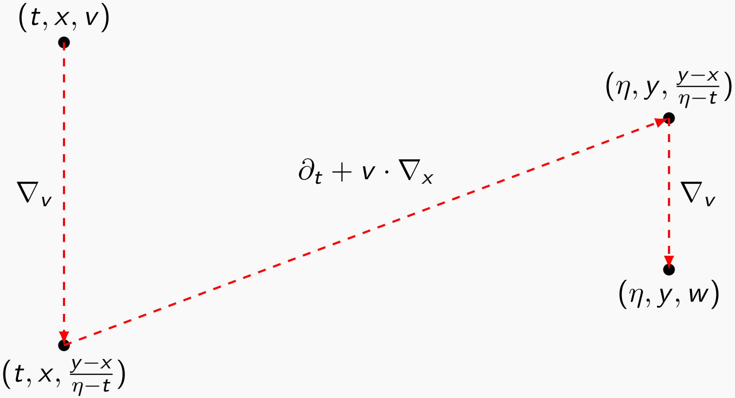

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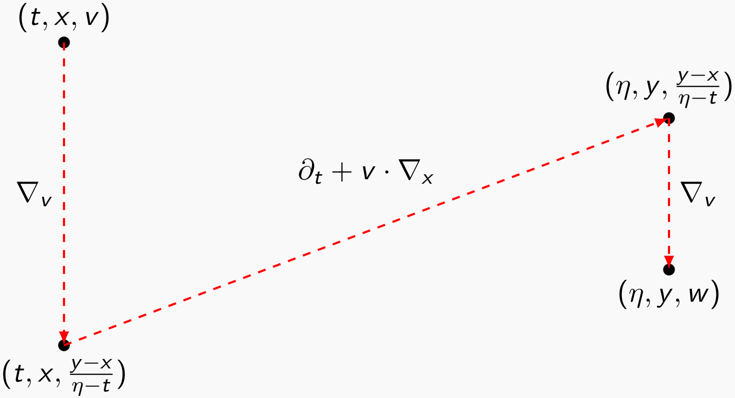


J. Guerand and C. Mouhot. *Quantitative De Giorgi methods in kinetic theory*, J. École polytech. - Math. **9** (2022), 1159-1181.

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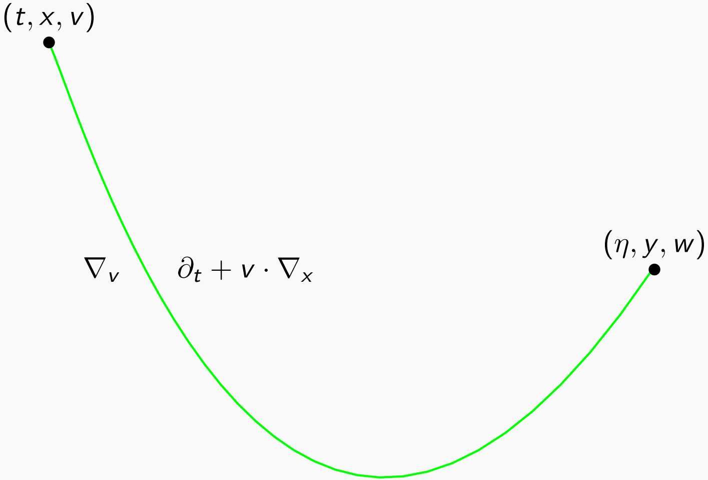

# Kinetic trajectories

Definition:

Let $(t, x, v)$ and $(\eta, y, w) \in \mathbb{R}^{1+2n}$ with $\eta \neq t$. A kinetic trajectory is a map

$$\gamma = \gamma(r) = \gamma(r; (t, x, v), (\eta, y, w)) = (\gamma_t(r), \gamma_x(r), \gamma_v(r)) \in \mathbb{R}^{1+2n}$$

defined for $r \in [0, 1]$ that is

- continuous on $r \in [0, 1]$ (and in particular bounded),
- differentiable on $r \in (0, 1)$,
- with endpoints $\gamma(0) = (t, x, v)$ and $\gamma(1) = (\eta, y, w)$,
- satisfying the constraint $\dot{\gamma}_x(r) = \dot{\gamma}_t(r)\gamma_v(r)$ for $r \in (0, 1)$.

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defined for $r \in [0, 1]$ that is sufficiently smooth

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- satisfying the constraint $\dot{\gamma}_x(r) = \dot{\gamma}_t(r)\gamma_v(r)$ for $r \in (0, 1)$.

For $g \colon \mathbb{R}^{1+2n} \to \mathbb{R}$ smooth

$$\frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) = \dot{\gamma}_t(r)[\partial_t g] + \dot{\gamma}_x(r) \cdot [\nabla_x g](\gamma(r)) + \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r))$$

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For $g\colon \mathbb{R}^{1+2n} \to \mathbb{R}$ smooth

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) &= \dot{\gamma}_t(r)[\partial_t g] + \dot{\gamma}_x(r) \cdot [\nabla_x g](\gamma(r)) + \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r)) \\ &= \dot{\gamma}_t(r)[\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r)).\end{aligned}$$

# Literature on trajectories

- Early works by Carathéodory 09, Rashevskii 38 and Chow 39.
- Breakthrough by Nagel, Stein and Wainger 85.
- Lots of works on Geometric Control theory.
- Trajectorial proof of Jerison's Poincaré inequality by Lanconelli-Morbidelli 00.
- Kinetic trajectories are constructed in Pascucci-Polidoro 04.

In none of these results $X_0$ and $X_1, \ldots, X_n$ are treated at the right scale.

# Critical kinetic trajectories

Today $\dot{\gamma}_t = \eta - t$.

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A kinetic trajectory is called a critical kinetic trajectory if it additionally satisfies

$$\left|\left(\nabla_{y,w}\gamma(r;(t,x,v),(\eta,y,w))^{-1}\right)_{\cdot;2}\right| \sim |\dot{\gamma}_v(r)| \sim r^{-\frac{1}{2}}$$

as $r \to 0$, $r \neq 0$.

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as $r \to 0$, $r \neq 0$.

Trajectories constructed in N.-Zacher 22 are not critical.

Neither are the ones in the follow-up work:

F. Anceschi, H. Dietert, J. Guerand, A. Loher, C. Mouhot, and A. Rebucci.
Poincaré inequality and quantitative De Giorgi method for hypoelliptic operators, 2024.

# Critical kinetic trajectories

Lemma (DMNZ 24):

There exists a family of critical kinetic trajectories given by

$$\gamma(r) = \begin{pmatrix} \gamma_t(r) \\ \gamma_x(r) \\ \gamma_v(r) \end{pmatrix} = \begin{pmatrix} t + (\eta - t)r \\ \mathcal{A}_{\eta-t}(r) \begin{pmatrix} y \\ w \end{pmatrix} + \mathcal{B}_{\eta-t}(r) \begin{pmatrix} x \\ v \end{pmatrix} \end{pmatrix}$$

with properties such as

- $\mathcal{A}_{\eta-t}(0) = 0$, $\mathcal{A}_{\eta-t}(1) = \mathrm{Id}_{2n}$ and $\mathcal{B}_{\eta-t}(0) = \mathrm{Id}_{2n}$, $\mathcal{B}_{\eta-t}(1) = 0$,
- $\det \mathcal{A}_{\eta-t}(r) = r^{2n}$, $\det \mathcal{B}_{\eta-t}(r) \approx (1-r)^{2n}$,
- spatial uniform control $\gamma(r) \in \tilde{Q}$,
- criticality, i.e. $|\dot{\gamma}_v| \lesssim r^{-\frac{1}{2}}$ and

$$\left| \left( \nabla_{y,w} \gamma(r; (t,x,v), (\eta, y, w))^{-1} \right)_{\cdot;2} \right| = \left| (\mathcal{A}^{-1}_{\eta-t})_{\cdot;2} \right| \lesssim r^{-\frac{1}{2}}.$$

# Construction of kinetic trajectories

Ansatz:

$$\dot\gamma_t = \eta - t \text{ and } \dot\gamma_v = \ddot g_0(r)\mathsf{m}_0 + \ddot g_1(r)\mathsf{m}_1$$

for two forcings $\ddot g_0, \ddot g_1 \colon [0,1] \to \mathbb{R}$ and vectorial parameters $\mathsf{m}_0, \mathsf{m}_1 \in \mathbb{R}^n$.

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Integration yields

$$\begin{cases} \dot{\gamma}_v(r) = \ddot{g}_0(r)\mathbf{m}_0 + \ddot{g}_1(r)\mathbf{m}_1 \\ \gamma_v(r) = \dot{g}_0(r)\mathbf{m}_0 + \dot{g}_1(r)\mathbf{m}_1 + v. \end{cases}$$

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A kinetic trajectory needs to satisfy

$$\dot\gamma_x(r) = \dot\gamma_t(r)\gamma_v(r) = (\eta - t)\dot g_0(r)\mathbf{m}_0 + (\eta - t)\dot g_1(r)\mathbf{m}_1 + (\eta - t)v$$

# Construction of kinetic trajectories

Ansatz:

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Endpoint condition determines the vectorial parameters

$$\begin{cases} \gamma_x(1) = (\eta - t)g_0(1)\mathbf{m}_0 + (\eta - t)g_1(1)\mathbf{m}_1 + (\eta - t)v + x = y \\ \gamma_v(1) = \dot g_0(1)\mathbf{m}_0 + \dot g_1(1)\mathbf{m}_1 + v = w \end{cases}$$

# Construction of kinetic trajectories

Ansatz:

$$\dot{\gamma}_t = \eta - t \text{ and } \dot{\gamma}_v = \ddot{g}_0(r)\mathrm{m}_0 + \ddot{g}_1(r)\mathrm{m}_1$$

for two forcings $\ddot{g}_0, \ddot{g}_1 \colon [0,1] \to \mathbb{R}$ and vectorial parameters $\mathrm{m}_0, \mathrm{m}_1 \in \mathbb{R}^n$.

Integration yields

$$\begin{cases} \gamma_v(r) = \dot{g}_0(r)\mathbf{m}_0 + \dot{g}_1(r)\mathbf{m}_1 + v \\ \gamma_x(r) = (\eta - t)g_0(r)\mathbf{m}_0 + (\eta - t)g_1(r)\mathbf{m}_1 + (\eta - t)rv + x \end{cases}$$

Endpoint condition determines the vectorial parameters

$$\begin{cases} (\eta - t)g_0(1)\mathbf{m}_0 + (\eta - t)g_1(1)\mathbf{m}_1 + (\eta - t)v + x = y \\ \dot{g}_0(1)\mathbf{m}_0 + \dot{g}_1(1)\mathbf{m}_1 + v = w \end{cases}$$

Criticality is achieved for a good choice of the forcing.

# Weak $L^1$-estimate for $\log f$



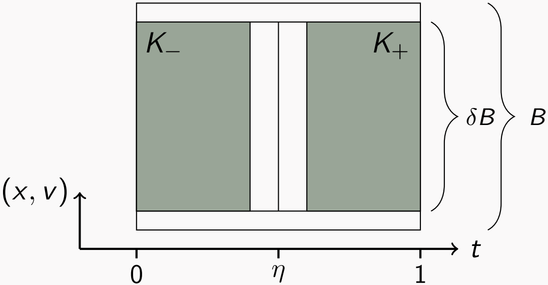
Theorem (DMNZ 24):

Let $\delta, \eta \in (0,1)$ and $\varepsilon > 0$. Then for any supersolution $f \geq \varepsilon > 0$ to (1) there exists a constant $C = C(n, \delta, \eta, \lambda, \Lambda) > 0$ such that

$$s \left| \{ (t,x,v) \in K_- \colon \log f(t,x,v) - c(f) > s \} \right| \leq C$$

$$s \left| \{ (t,x,v) \in K_+ \colon c(f) - \log f(t,x,v) > s \} \right| \leq C$$

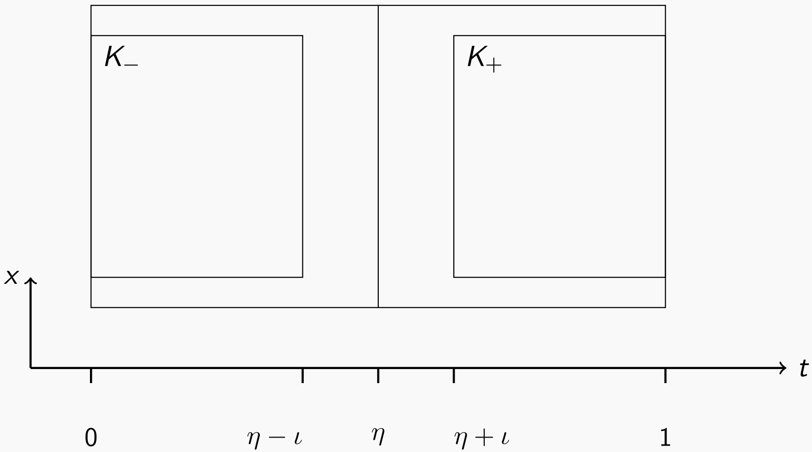
for all $s > 0$ with $c(f) = \dfrac{1}{c_\varphi} \displaystyle\int_B \log f(\eta, y, w) \varphi^2(y,w) \mathrm{d}(y,w)$.



# Proof of the weak $L^1$-estimate

Unit size. $\mathfrak{a} = \mathrm{Id}$ for simplicity. Goal:

$$s\,|\{(t,x,v) \in K_- :\ \log f(t,x,v) - c(f) > s\}| \le C, \quad s > 0$$



# Proof of the weak $L^1$-estimate

Recall

$$c(f) = \frac{1}{c_\varphi} \int\limits_B [\log f](\eta, y, w)\varphi^2(y, w)\mathrm{d}(y, w).$$

where

$$c_\varphi = \int_B \varphi^2(y, w)\mathrm{d}(y, w).$$

# Proof of the weak $L^1$-estimate

Recall

$$c(f) = \frac{1}{c_\varphi} \int_B [\log f](\eta, y, w)\varphi^2(y, w)\mathrm{d}(y, w).$$

Note that

$$\begin{aligned} & s \left|\{(t, x, v) \in K_- \colon \log(f) - c(f) > s\}\right| \\ & \leq \int_0^{\eta-\iota} \int_B ([\log f](t, x, v) - c(f))_+ \mathrm{d}(t, x, v) \end{aligned}$$

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Recall

$$c(f) = \frac{1}{c_\varphi} \int\limits_B [\log f](\eta, y, w)\varphi^2(y, w)\mathrm{d}(y, w).$$

Goal: estimate

$$\int\limits_0^{\eta-\iota} \int\limits_B ([\log f](t, x, v) - c(f))_+ \mathrm{d}(t, x, v) \leq C$$

by a constant.

$L^1$-Poincaré inequality in spacetime without a gradient.

# Proof of the $L^1$-estimate



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by a constant.

$L^1$-Poincaré inequality in spacetime without a gradient.

Recall: if $f$ is supersolution to (1), then $g = \log f$ is a supersolution to

$$\partial_t g + v \cdot \nabla_x g = \Delta_v g + |\nabla_v g|^2$$

# Proof of the $L^1$-estimate



For $g = \log f$ we have

$$g(t,x,v) - c(f)$$
$$= \frac{1}{c_\varphi} \int_B \left(g(t,x,v) - g(\eta,y,w)\right)\varphi^2(y,w)\mathrm{d}(y,w)$$
$$= -\frac{1}{c_\varphi} \int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r))\mathrm{d}r \; \varphi^2(y,w)\mathrm{d}(y,w)$$

# Proof of the $L^1$-estimate

$(1)\ \partial_t g + v \cdot \nabla_x g = \Delta_v g + |\nabla_v g|^2$

For $g = \log f$ we have

$$
\begin{aligned}
&g(t,x,v) - c(f) \\
&= \frac{1}{c_\varphi} \int_B \left(g(t,x,v) - g(\eta,y,w)\right) \varphi^2(y,w)\mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r))\mathrm{d}r \ \varphi^2(y,w)\mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \dot{\gamma}_t(r)[\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r))\mathrm{d}r \ \varphi^2 \mathrm{d}(y,w)
\end{aligned}
$$

# Proof of the $L^1$-estimate



For $g = \log f$ we have

$$
\begin{aligned}
& g(t,x,v) - c(f) \\
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&= -\frac{1}{c_\varphi} \int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) \mathrm{d}r \ \varphi^2(y,w) \mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \dot\gamma_t(r) [\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot\gamma_v(r) \cdot [\nabla_v g](\gamma(r)) \mathrm{d}r \ \varphi^2 \mathrm{d}(y,w) \\
&\leq -\frac{\eta - t}{c_\varphi} \int_B \int_0^1 [\Delta_v g](\gamma(r)) + |\nabla_v g|^2 (\gamma(r)) \mathrm{d}r \ \varphi^2(y,w) \mathrm{d}(y,w) \\
&\quad - \frac{1}{c_\varphi} \int_B \int_0^1 \dot\gamma_v(r) \cdot [\nabla_v g](\gamma(r)) \mathrm{d}r \ \varphi^2(y,w) \mathrm{d}(y,w)
\end{aligned}
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# Proof of the $L^1$-estimate

$(1)\ \partial_t g + v \cdot \nabla_x g = \Delta_v g + |\nabla_v g|^2$

For $g = \log f$ we have

$$
\begin{aligned}
&g(t,x,v) - c(f) \\
&= \frac{1}{c_\varphi} \int_B \left( g(t,x,v) - g(\eta,y,w) \right) \varphi^2(y,w) \mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) \mathrm{d}r\ \varphi^2(y,w) \mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \dot\gamma_t(r) [\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot\gamma_v(r) \cdot [\nabla_v g](\gamma(r)) \mathrm{d}r\ \varphi^2 \mathrm{d}(y,w) \\
&\le -\frac{\eta - t}{c_\varphi} \int_B \int_0^1 [\Delta_v g](\gamma(r)) + |\nabla_v g|^2 (\gamma(r)) \mathrm{d}r\ \varphi^2(y,w) \mathrm{d}(y,w) \\
&\quad - \frac{1}{c_\varphi} \int_B \int_0^1 \dot\gamma_v(r) \cdot [\nabla_v g](\gamma(r)) \mathrm{d}r\ \varphi^2(y,w) \mathrm{d}(y,w)
\end{aligned}
$$

Idea: use quadratic gradient term to absorb all gradients

# The forcing terms

Recall that $|\dot{\gamma}_v| \lesssim r^{-\frac{1}{2}}$, hence

$$-\frac{1}{c_\varphi}\int_B\int_0^1 \dot{\gamma}_v(r)\cdot[\nabla_v g](\gamma(r))\mathrm{d}r\ \varphi^2(y,w)\mathrm{d}(y,w)$$

$$\lesssim \int_B\int_0^1 r^{-\frac{1}{2}}\left|\nabla_v g\right|(\gamma(r))\mathrm{d}r\ \varphi(y,w)\mathrm{d}(y,w)$$

# Partial integration



$$\int_B [\Delta_v g](\gamma(r))\varphi^2(y,w)\mathrm{d}(y,w)$$

# Partial integration



Substitute $(\tilde{y}, \tilde{w}) = \Phi(y, w) = \Phi_{r,t,x,v,\eta}(y, w) := (\gamma_x(r), \gamma_v(r))$.

$$\int_B [\Delta_v g](\gamma(r))\varphi^2(y, w)\mathrm{d}(y, w)$$
$$= \int_{\Phi(B)} [\Delta_v g](\gamma_t(r), \tilde{y}, \tilde{w})\varphi^2(\Phi^{-1}(\tilde{y}, \tilde{w}))\left|\det \mathcal{A}\right|^{-1} \mathrm{d}(\tilde{y}, \tilde{w})$$

# Partial integration



Substitute $(\tilde{y}, \tilde{w}) = \Phi(y, w) = \Phi_{r,t,x,v,\eta}(y, w) := (\gamma_x(r), \gamma_v(r))$.

$$\begin{aligned}
&\int_B [\Delta_v g](\gamma(r))\varphi^2(y, w)\mathrm{d}(y, w) \\
&= \int_{\Phi(B)} [\Delta_v g](\gamma_t(r), \tilde{y}, \tilde{w})\varphi^2(\Phi^{-1}(\tilde{y}, \tilde{w}))\, |\det \mathcal{A}|^{-1}\, \mathrm{d}(\tilde{y}, \tilde{w}) \\
&= -\int_{\Phi(B)} \langle [\nabla_v g](\gamma_t(r), \tilde{y}, \tilde{w}), \nabla_v \varphi^2(\Phi^{-1}(\tilde{y}, \tilde{w}))\rangle\, |\det \mathcal{A}|^{-1}\, \mathrm{d}(\tilde{y}, \tilde{w}) \\
&= -2\int_{\Phi(B)} \langle [\nabla_v g](\gamma_t(r), \tilde{y}, \tilde{w}), [\nabla\varphi]^T(\Phi^{-1}(\tilde{y}, \tilde{w}))(\mathcal{A}(r)^{-1})_{\cdot 2}\rangle \\
&\qquad\qquad \varphi(\Phi^{-1}(\tilde{y}, \tilde{w})) \cdot |\det \mathcal{A}|^{-1}\, \mathrm{d}(\tilde{y}, \tilde{w})
\end{aligned}$$

# Partial integration



Substitute $(\tilde{y}, \tilde{w}) = \Phi(y, w) = \Phi_{r,t,x,v,\eta}(y, w) := (\gamma_x(r), \gamma_v(r))$.

$$
\begin{aligned}
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&= \int_{\Phi(B)} [\Delta_v g](\gamma_t(r), \tilde{y}, \tilde{w})\varphi^2(\Phi^{-1}(\tilde{y}, \tilde{w}))\, |\det \mathcal{A}|^{-1}\, \mathrm{d}(\tilde{y}, \tilde{w}) \\
&= -\int_{\Phi(B)} \langle [\nabla_v g](\gamma_t(r), \tilde{y}, \tilde{w}), \nabla_v \varphi^2(\Phi^{-1}(\tilde{y}, \tilde{w}))\rangle\, |\det \mathcal{A}|^{-1}\, \mathrm{d}(\tilde{y}, \tilde{w}) \\
&= -2\int_{\Phi(B)} \langle [\nabla_v g](\gamma_t(r), \tilde{y}, \tilde{w}), [\nabla\varphi]^T(\Phi^{-1}(\tilde{y}, \tilde{w}))(\mathcal{A}(r)^{-1})_{\cdot 2}\rangle \\
&\qquad\qquad \varphi(\Phi^{-1}(\tilde{y}, \tilde{w})) \cdot |\det \mathcal{A}|^{-1}\, \mathrm{d}(\tilde{y}, \tilde{w}) \\
&= -2\int_B \langle [\nabla_v g](\gamma(r)), [\nabla\varphi]^T(y, w)(\mathcal{A}(r)^{-1})_{\cdot 2}\rangle \varphi(y, w)\mathrm{d}(y, w) \\
&\lesssim r^{-1/2} \int_B |\nabla_v g|\, (\gamma(r))\varphi(y, w)\mathrm{d}(y, w),
\end{aligned}
$$

# Conclusion

We obtain

$$(g(t,x) - c(f))_+$$
$$\lesssim \int_0^1 \int_B \Big( M r^{-1/2} \left|\nabla_v g\right| (\gamma(r))\varphi(y,w) - \left|\nabla_v g\right|^2 (\gamma(r))\varphi^2(y,w) \Big)_+ \mathrm{d}(y,w)\mathrm{d}r$$

for some constant $M > 0$.

# Conclusion

We obtain

$$(g(t,x) - c(f))_+$$
$$\lesssim \int_0^1 \int_B \Big( Mr^{-1/2} \left|\nabla_v g\right| (\gamma(r))\varphi(y,w) - \left|\nabla_v g\right|^2 (\gamma(r))\varphi^2(y,w) \Big)_+ \mathrm{d}(y,w)\mathrm{d}r$$

for some constant $M > 0$.

Integrate on $K_-$ and substitute $(\tilde{t},\tilde{x},\tilde{v}) = \gamma(r)$ for $r \approx 0$.

Calculating the $r$-integral from 0 to $\min\{1/2, M^2/p^2\}$ yields

$$\int_0^{1/2} \Big( r^{-1/2} Mp - p^2 \Big)_+ \mathrm{d}r \lesssim M^2$$

for all $p > 0$. Here $p = \left|\nabla_v g\right| (\tilde{t},\tilde{x},\tilde{v})\varphi(y,w)$.

# Conclusion

We obtain

$$
\begin{aligned}
&(g(t,x) - c(f))_+ \\
&\lesssim \int_0^1 \int_B \Big( M r^{-1/2} \left| \nabla_v g \right| (\gamma(r)) \varphi(y,w) - \left| \nabla_v g \right|^2 (\gamma(r)) \varphi^2(y,w) \Big)_+ \mathrm{d}(y,w) \mathrm{d}r
\end{aligned}
$$

for some constant $M > 0$.

Integrate on $K_-$ and substitute $(\tilde{t}, \tilde{x}, \tilde{v}) = \gamma(r)$ for $r \approx 0$.

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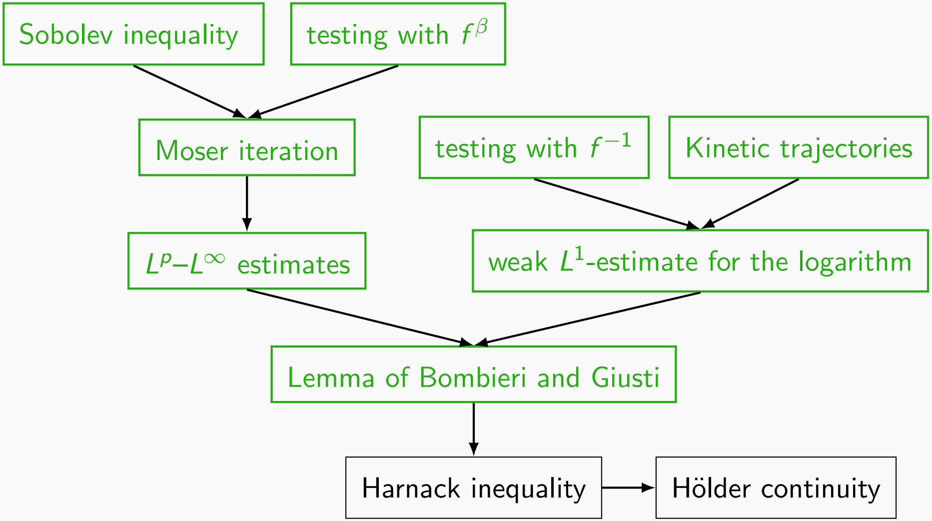
for all $p > 0$. Here $p = \left| \nabla_v g \right| (\tilde{t}, \tilde{x}, \tilde{v}) \varphi(y,w)$.

# Moser's 1971 method in kinetic theory



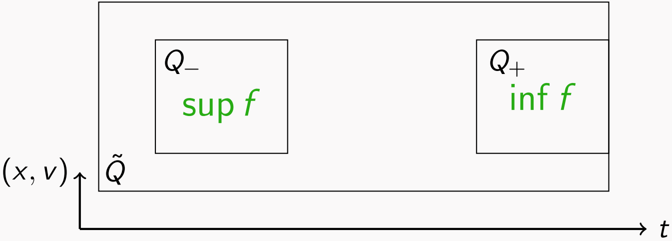
# Harnack inequality



Theorem (DMNZ 24):

There exists a universal const $C = C(n, \lambda, \Lambda) > 0$ such that for any nonnegative weak solution $f$ of (1) in $\tilde{Q}$ we have

$$\sup_{Q_-} f \le C \inf_{Q_+} f.$$



# Harnack inequality

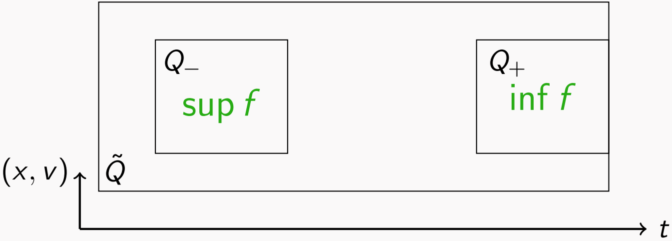


Theorem (DMNZ 24):

There exists a universal const $C = C(n) > 0$ such that for any nonnegative weak solution $f$ of (1) in $\tilde{Q}$ we have

$$\sup_{Q_-} f \leq C^{\mu} \inf_{Q_+} f.$$

Here, $\mu = \frac{1}{\lambda} + \Lambda$ if $\mathfrak{a}$ is symmetric. Optimal!



# Weak Harnack inequality

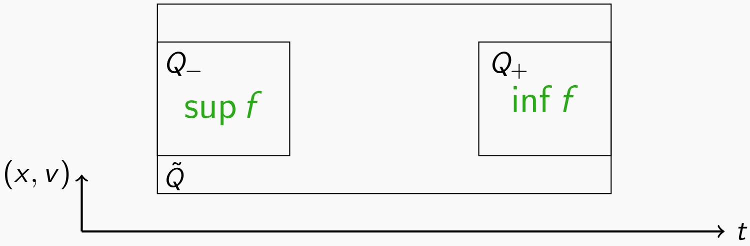


Theorem (DMNZ 24):

There exists a universal $C(n,\mu) > 0$ such that for all $p \in (0, 1 + \frac{1}{2n})$ and any nonnegative weak supersolution $f$ to (1) in $\tilde{Q}$ we have

$$\left( \int_{Q_-} |f|^p \, \mathrm{d}(t,x,v) \right)^p \leq C \inf_{Q_+} f.$$

Optimal range for $p$.



# Euclidean smoothing

$$f = f(v) \mapsto \int_{\mathbb{R}^n} f(\mathsf{m})\varphi^2\left(\frac{v-\mathsf{m}}{r}\right) r^{-n}\mathrm{d}\mathsf{m} = \int_{\mathbb{R}^n} f(v - r\mathsf{m})\varphi^2(\mathsf{m})\mathrm{d}\mathsf{m}$$

# Parabolic smoothing

Space

$$f = f(t, v) \mapsto \int_{\mathbb{R}^n} f(t - sr, v - r^{1/2}\mathsf{m})\varphi^2(\mathsf{m})\mathrm{d}\mathsf{m}$$

Spacetime

$$f = f(t, v) \mapsto \int_{\mathbb{R}^{1+n}} f(t - sr, v - r^{1/2}\mathsf{m})\psi^2(s, \mathsf{m})\mathrm{d}(s, \mathsf{m})$$

# Kinetic smoothing

Consider $\gamma^{(s,\mathsf{m})}\colon \mathbb{R} \to \mathbb{R}^{1+2n}$ with $\mathsf{m} = (\mathsf{m}_0, \mathsf{m}_1) \in \mathbb{R}^{2n}$, $s \neq 0$ defined as

$$\gamma^{(s,\mathsf{m})}(r; (t,x,v)) = \begin{pmatrix} \gamma_t^{(s,\mathsf{m})}(r) \\ \gamma_x^{(s,\mathsf{m})}(r) \\ \gamma_v^{(s,\mathsf{m})}(r) \end{pmatrix} = \begin{pmatrix} t + s\,r \\ \mathcal{A}_s(r) \begin{pmatrix} \mathsf{m}_0 \\ \mathsf{m}_1 \end{pmatrix} + \begin{pmatrix} 1 & s\,r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \end{pmatrix}$$

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Space

$$[S_r(f)](t,x,v) = \frac{1}{c_\varphi}\int_B f(\gamma^{(s,\mathrm{m})}(r;(t,x,v))\varphi^2(\mathrm{m})\mathrm{d}\mathrm{m}$$

Spacetime

$$[T_r(f)](t,x,v) = \frac{1}{c_\psi}\int_Q f(\gamma^{(s,\mathrm{m})}(r;(t,x,v))\psi^2(s,\mathrm{m})\mathrm{d}(s,\mathrm{m})$$

# Kinetic Sobolev embedding

Theorem (DMNZ 24):

Let $f \in L^2(\mathbb{R}^{1+n}; \dot{H}^1(\mathbb{R}^n))$ such that $\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot h$ for some $h \in L^2(\mathbb{R}^{1+2n}; \mathbb{R}^n)$, then

$$\|f\|_{L^{2\kappa}(\mathbb{R}^{1+2n})} \leq C \left( \|\nabla_v f\|_{L^2(\mathbb{R}^{1+2n})} + \|h\|_{L^2(\mathbb{R}^{1+2n})} \right)$$

with $\kappa = 1 + \frac{1}{2n}$ and $C = C(n) > 0$.

# Kinetic Nash inequality

Theorem (DMNZ 24):

Let $f \in L^2(\mathbb{R}^{1+n}; \dot{H}^1(\mathbb{R}^n)) \cap L^1(\mathbb{R}^{1+2n})$ such that we have $\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot h$ for some $h \in L^2(\mathbb{R}^{1+2n}; \mathbb{R}^n)$, then

$$\|f\|_{L^2(\mathbb{R}^{1+2n})}^{1+\frac{2}{2+4d}} \leq C\sqrt{\|\nabla_v f\|_{L^2(\mathbb{R}^{1+2n})}^2 + \|h\|_{L^2(\mathbb{R}^{1+2n})}^2}\,\|f\|_{L^1(\mathbb{R}^{1+2n})}^{\frac{2}{2+4d}}$$

for some $C = C(n) > 0$.

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