Trajectories and De Giorgi-Nash-Moser theory

Lukas Niebel, joint work with Rico Zacher
Applied Mathematics - University Münster

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Parabolic diffusion problem

Let $\Omega \subset \mathbb{R}^d$ open and $T > 0$. Consider weak solutions $u = u(t,x) \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$ to

$$\partial_t u = \nabla \cdot (A \nabla u) + b \cdot \nabla u + cu + f \quad \text{in } (0,T) \times \Omega$$

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Parabolic diffusion problem with rough coefficients

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where $A = A(t,x) \colon (0,T) \times \Omega \to \mathbb{R}^{n \times n}$ is measurable, symmetric, bounded and

$$\lambda \left|\xi\right|^2 \leq \langle A(t,x)\xi, \xi \rangle \leq \Lambda \left|\xi\right|^2$$

for all $\xi \in \mathbb{R}^n$ a.e. $(t,x) \in (0,T) \times \Omega$. Set $\mu = \frac{1}{\lambda} + \Lambda$.

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that is

$$\int_0^T \int_\Omega -\partial_t \varphi(t, x) u(t, x) + \langle A(t, x) \nabla u(t, x), \nabla \varphi(t, x) \rangle \mathrm{d}x \mathrm{d}t = 0$$

for all $\varphi \in H^1((0, T); L^2(\Omega)) \cap L^2((0; T); \dot{H}^1(\Omega))$
with $\varphi|_{t=0} = 0 = \varphi|_{t=T}$.

Parabolic geometry

Consider $A = \mathrm{Id}$.

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if u solves (1) then so does $w(t,x) = u(\lambda^2 t, \lambda x)$

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Parabolic cylinders: $Q_r(t_0, x_0) = (t_0 - r^2, t_0 + r^2) \times B_r(x_0)$

A priori estimates

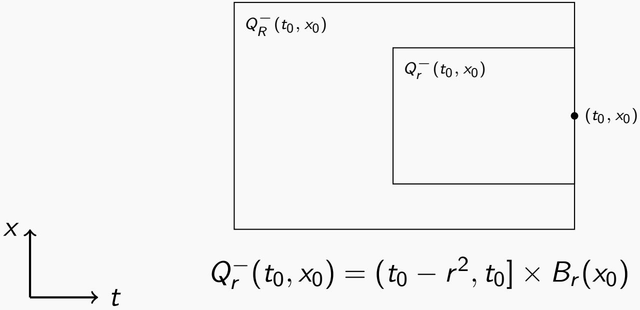
1. Local boundedness
2. The Harnack inequality
3. Hölder continuity

$L^2 - L^\infty$ estimate

Theorem (Moser 64):

Let $\delta \in (0,1)$, $\delta \leq r < R \leq 1$, $t_0 \in (0,T)$, $x_0 \in \Omega$. There exists $c = c(d,\delta,\mu) > 0$ such that any pos. subsolution to (1) satisfies

$$\sup_{Q_r^-(t_0,x_0)} u^2 \leq \frac{c}{(R-r)^{d+2}} \iint_{Q_R^-(t_0,x_0)} u^2 \mathrm{d}x\mathrm{d}t$$



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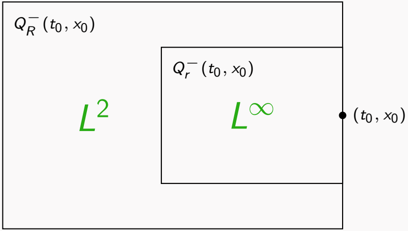

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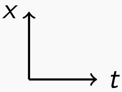
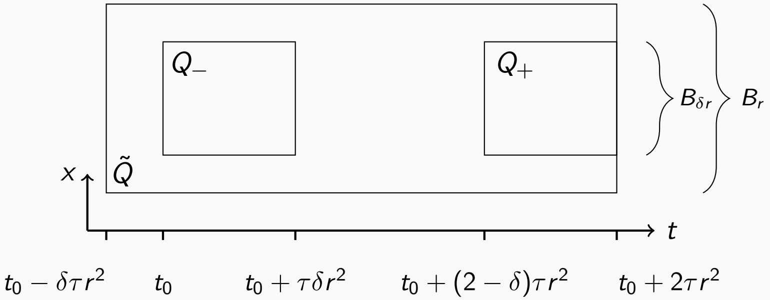


The Harnack inequality

Theorem (Moser 64):

Let $\delta \in (0,1)$, $\tau > 0$. There exists $C = C(d, \delta, \tau) > 0$ such that for any nonnegative weak solution u of (1) in $\tilde{Q}$ we have

$$\sup_{Q_-} u \leq C^{\mu} \inf_{Q_+} u.$$



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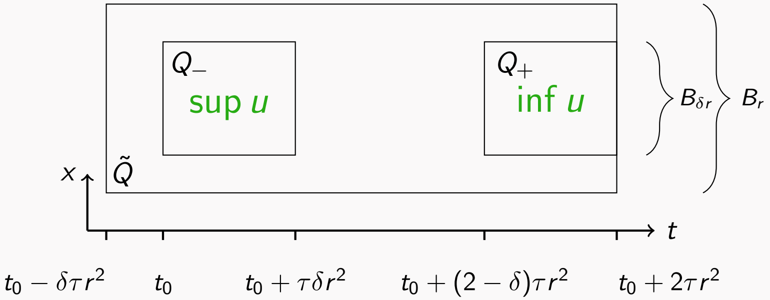

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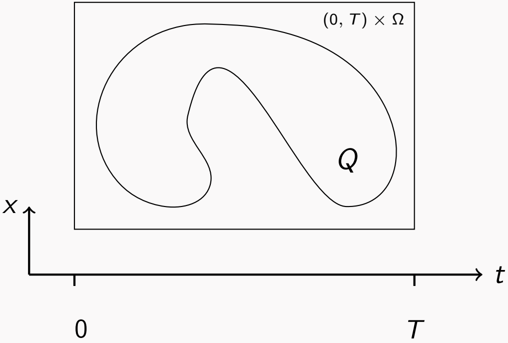
- scaling and translation invariant
- implies Hölder continuity in (t, x) of u
- implies heat kernel bounds
- dependency of the constant on $\mu = \frac{1}{\lambda} + \Lambda$ is optimal

Hölder continuity

Theorem (Nash 58, Moser 64):

Let u be a weak solution to (1) and $Q \subset\subset (0, T) \times \Omega$.
Then there exists $\varepsilon, C > 0$ such that $u \in C^{\varepsilon}(\bar{Q})$ and

$$\|u\|_{C^{\varepsilon}(\bar{Q})} \leq C \|u\|_{L^2((0,T)\times\Omega)}.$$



Brief history

- Harnack proves inequality for harmonic functions $\Delta u = 0$ in 1887
- Hadamard & Pini independently prove a Harnack inequality for the heat equation $\partial_t u = \Delta u$ in '57
- De Giorgi solves Hilbert's 19th problem in '57 key step: a priori Hölder continuity for $-\nabla \cdot (A \nabla u) = 0$
- '58 Nash treats the elliptic & parabolic eq. $\partial_t u = \nabla \cdot (A \nabla u)$

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- Moser proves Harnack inequality for the elliptic problem '61
- Moser proves Harnack inequality for the parabolic problem '64
- Moser provides a simpler proof of the Harnack inequality in '71 based on ideas due to Bombieri and Giusti
- Bombieri and Giusti prove a Harnack inequality for elliptic differential equations on minimal surfaces in '72

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Proof of the Harnack inequality à la Moser '71

3 Ingredients:

A: $L^p - L^\infty$ estimate for small $p \neq 0$

B: weak L^1-estimate for the logarithm of supersolutions

C: Lemma of Bombieri and Giusti

$L^p - L^\infty$ estimate

<u>Theorem (Moser 64 & 71):</u>

Let $\delta \in (0,1)$, $\delta \le r < R \le 1$, $t_0 \in (0,T)$, $x_0 \in \Omega$. There exists $c = c(d,\delta) > 0$ such that any pos. solution to (1) satisfies

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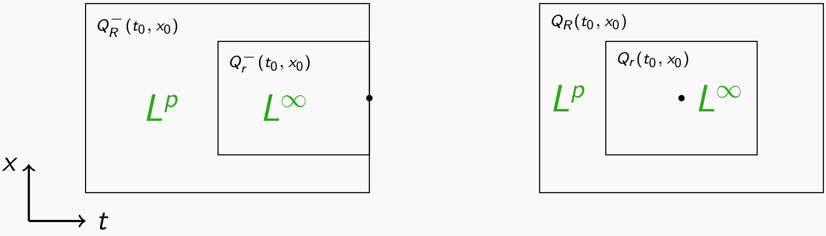

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- test the equation (1) with $u^\beta \varphi^2$, $\beta \in (-\infty, -1)$
- employ the Sobolev inequality to obtain a gain of integrability on smaller cylinder
- iterate this inequality (Moser iteration)

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Weak L^1-estimate for $\log u$

Theorem (Moser 64 & 71):

Let $\delta, \eta \in (0,1)$ and $\varepsilon, \tau > 0$. Then for any supersolution $u \geq \varepsilon > 0$ to (1) there exists constants $c = c(u)$ and $C = C(d, \delta, \eta, \tau) > 0$ s.t.

$$s \left|\{(t,x) \in K_- \colon \log u(t,x) - c(u) > s\}\right| \leq C \mu r^2 \left|B\right|, \quad s > 0$$

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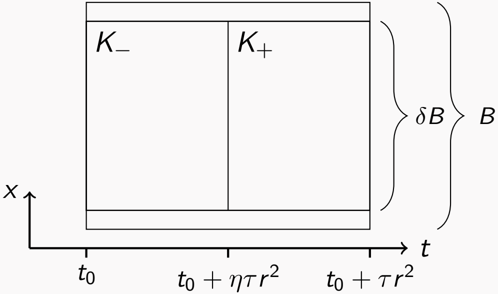

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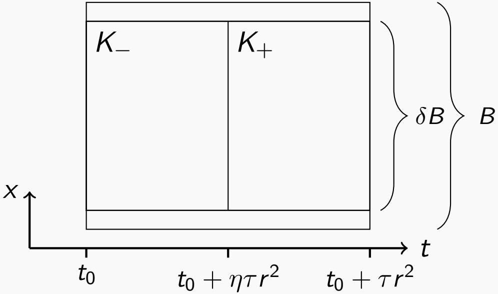

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Idea of the proof:

- if u is a supersolution to (1), then *logu* is a supersolution to

$$\partial_t \log u = \nabla \cdot (A \nabla \log u) + \langle A \nabla \log u, \nabla \log u \rangle$$

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Idea of the proof:

- test the equation with $u^{-1}\varphi^2$
- employ the spatial Poincaré inequality to obtain a differential inequality for

$$t \mapsto W(t) = \int_B \log u(t,y) \varphi^2(y) \mathrm{d}y$$

- several clever estimations yield the statement

Lemma of Bombieri and Giusti

Lemma (Moser 71, Bombieri and Giusti 72):

Let (X, ν) be a finite measure space, $U_\sigma \subset X$, $0 < \sigma \leq 1$ measurable with $U_{\sigma'} \subset U_\sigma$ if $\sigma' \leq \sigma$. Let $C_1, C_2 > 0$, $\delta \in (0,1)$, $\tilde{\mu} > 1$, $\gamma > 0$. Suppose $0 \leq f \colon U_1 \to \mathbb{R}$ satisfies the following two conditions:

- for all $0 < \delta \leq r < R \leq 1$ and $0 < p < 1/\tilde{\mu}$ we have

$$\sup_{U_r} f^p \leq \frac{C_1}{(R-r)^\gamma \nu(U_1)} \int_{U_R} f^p \mathrm{d}\nu$$

- $s\nu(\{\log f > s\}) \leq C_2 \tilde{\mu}\ \nu(U_1)$ for all $s > 0$.

Then

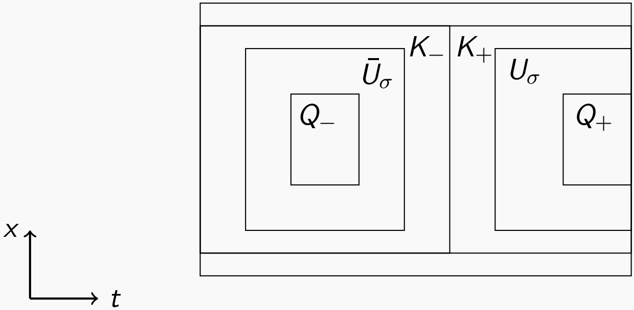
$$\sup_{U_\delta} f \leq C^{\tilde{\mu}}$$

where $C = C(C_1, C_2, \delta, \gamma)$.

Proof of the Harnack à la Moser '71

Goal:

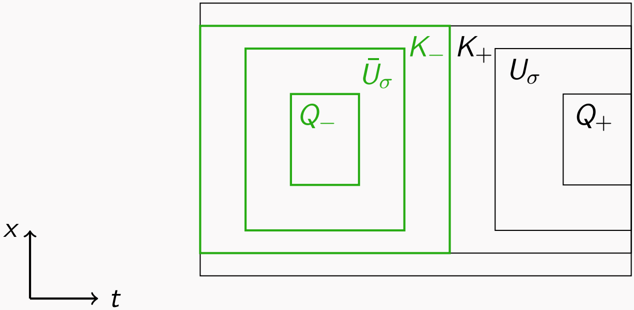
$$\sup_{Q_-} u \le C^{\mu} \inf_{Q_+} u.$$



Proof of the Harnack inequality à la Moser '71

Consider first $u\exp(-c(u))$ with $c(u)$ as in weak L^1-estimate.
Then the A,B and C combined give

$$\sup_{Q_-} u \leq e^{c(u)} \exp\left(C\mu\right)$$



Proof of the Harnack inequality à la Moser '71

Consider now $u^{-1}\exp(c(u))$ with $c(u)$ as in weak L^1-estimate. Then the A,B and C combined give

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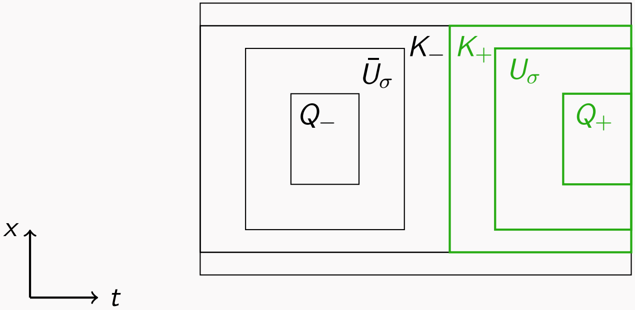

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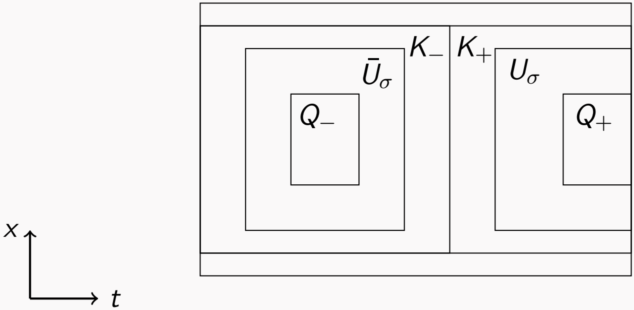
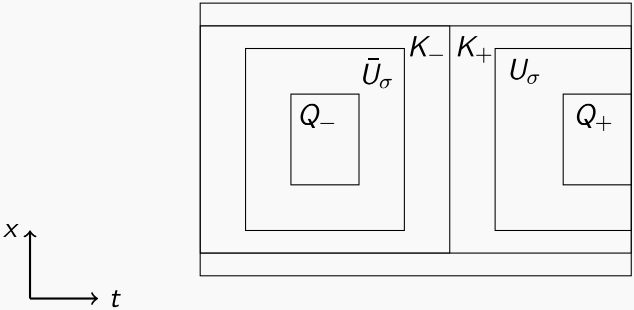


Proof of the Harnack inequality à la Moser '71

$$\left.\begin{aligned} e^{c(u)} &\leq \exp(C\mu)\inf_{Q_+} u \\ \sup_{Q_-} u &\leq e^{c(u)}\exp(C\mu) \end{aligned}\right\} \Rightarrow \text{Harnack inequality}$$



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  the method is less technical
- very robust
- allows to obtain the optimal dependency of the constants on λ, Λ
- one can also include source terms or lower order terms

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- allows to obtain the optimal dependency of the constants on λ, Λ
- one can also include source terms or lower order terms
- can be applied in many other contexts
  - a class of hypoelliptic equations (type A) (Lu '92)
  - discrete space problems (Delmotte '99)
  - fractional (in time) equations (Zacher '13)
  - non-local (in space) equations (Kassmann & Felsinger '13)
  - passive scalars with rough drifts (Albritton & Dong '22)
  - many more

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Problem (e.g. for the application to kinetic equations):

The weak L^1-estimate heavily relies on a spatial Poincaré inequality.

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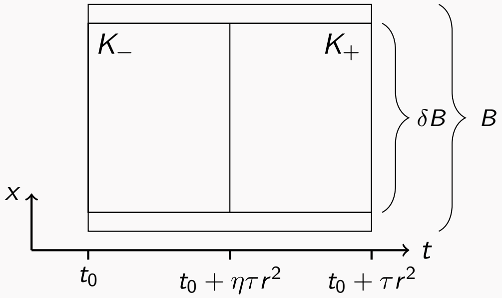

Weak L^1-estimate for $\log u$ modified

$(1)\ \partial_t u = \nabla \cdot (A \nabla u)$

Theorem (Moser 64 & 71, N. & Zacher 22):

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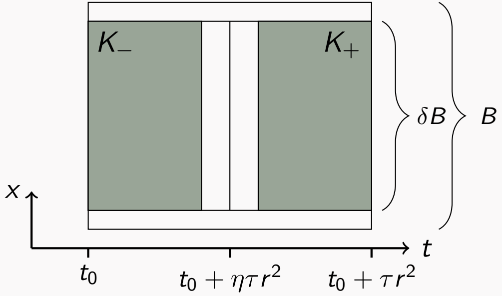
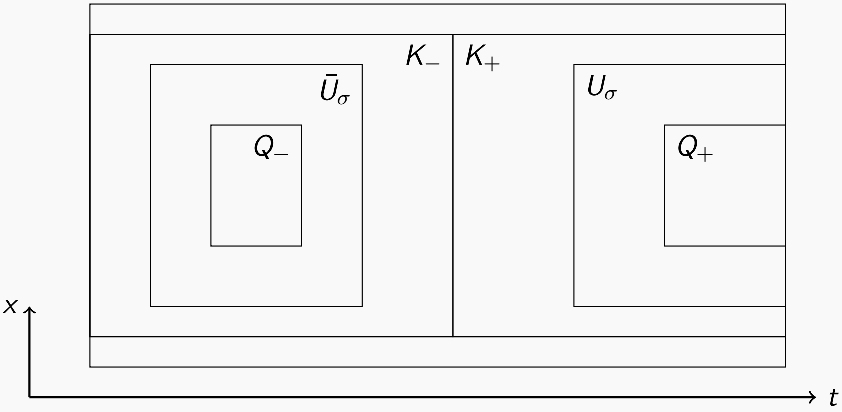


Proof of the Harnack inequality à la Moser '71



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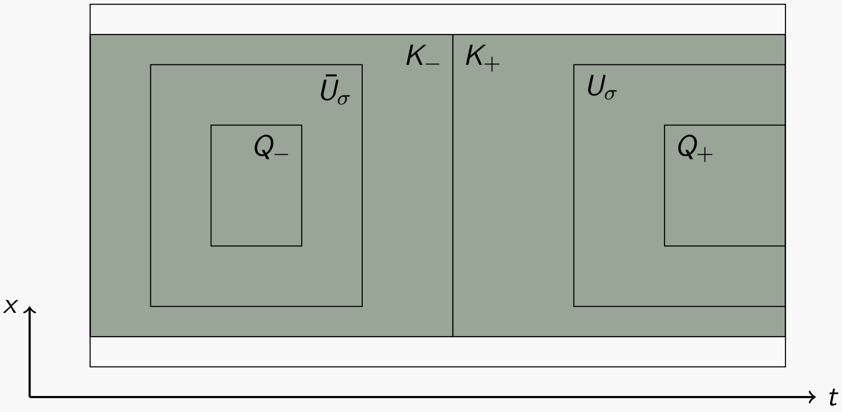
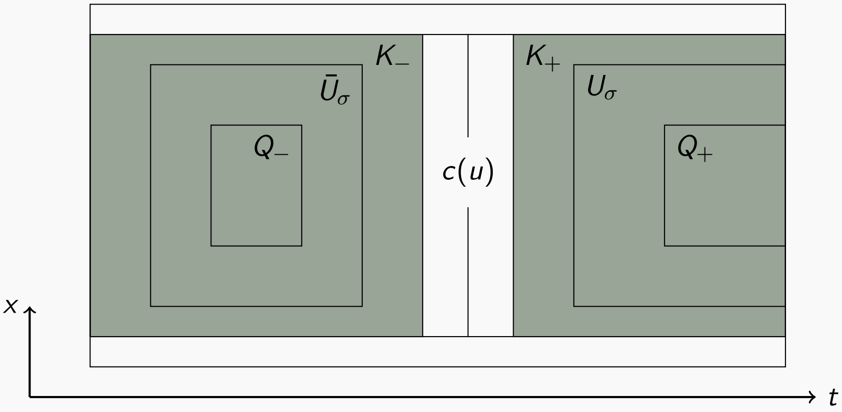


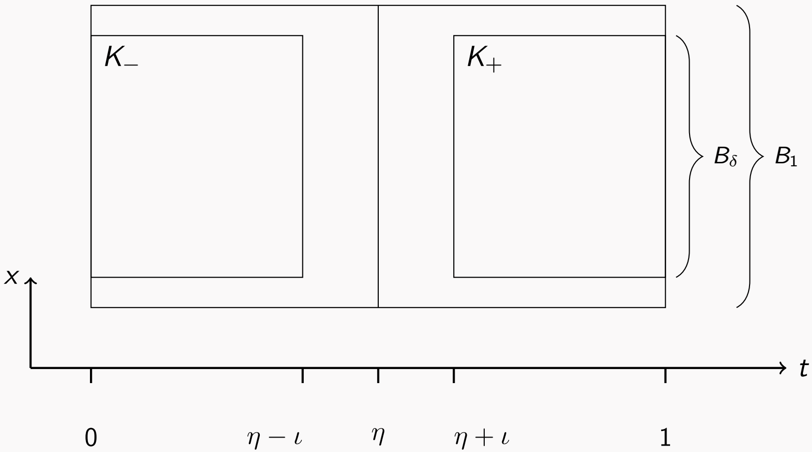
Proof of the Harnack inequality à la Moser '71 modified



Proof of the weak L^1-estimate mod.

By scaling and translation $t_0 = 0$, $r = 1$. $\tau = 1$ for simplicity.

$$s \left|\{(t,x) \in K_- \colon \log u(t,x) - c(u) > s\}\right| \le C\mu r^2 \left|B\right|, \quad s > 0$$



Proof of the weak L^1-estimate mod.

Choose

$$c(u) = \frac{1}{c_\varphi} \int\limits_B [\log u](\eta, y)\varphi^2(y)\mathrm{d}y$$

where

$$c_\varphi = \int_B \varphi^2(y)\mathrm{d}y.$$

Proof of the weak L^1-estimate mod.

Choose

$$c(u) = \frac{1}{c_\varphi} \int_B [\log u](\eta, y)\varphi^2(y)\mathrm{d}y.$$

Note that

$$s\,|\{(t,x) \in K_- \colon \log(u) - c(u) > s\}| \le \int_0^{\eta-\iota} \int_B ([\log u](t,x) - c(u))_+ \mathrm{d}x\mathrm{d}t$$

Proof of the weak L^1-estimate mod.

Choose

$$c(u) = \frac{1}{c_\varphi} \int_B [\log u](\eta, y)\varphi^2(y)\mathrm{d}y$$

Note that

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Proof of the weak L^1-estimate mod.

Choose

$$c(u) = \frac{1}{c_\varphi} \int_B [\log u](\eta, y)\varphi^2(y)\mathrm{d}y$$

Goal: estimate

$$\int_0^{\eta-\iota} \int_B ([\log u](t,x) - c(u))_+ \mathrm{d}x\mathrm{d}t$$

by a constant

L^1-Poincaré inequality in space time without gradient?!

Proof of the weak L^1-estimate mod.

Choose

$$c(u) = \frac{1}{c_\varphi} \int_B [\log u](\eta, y) \varphi^2(y) \mathrm{d}y$$

Goal: estimate

$$\int_0^{\eta-\iota} \int_B ([\log u](t,x) - c(u))_+ \mathrm{d}x \mathrm{d}t$$

by a constant

L^1-Poincaré inequality in space time without gradient?!

Recall: if u is solution to (1), then $g = \log u$ is a super solution to

$$\partial_t g = \nabla \cdot (A \nabla g) + \langle A \nabla g, \nabla g \rangle.$$

Proof using trajectories

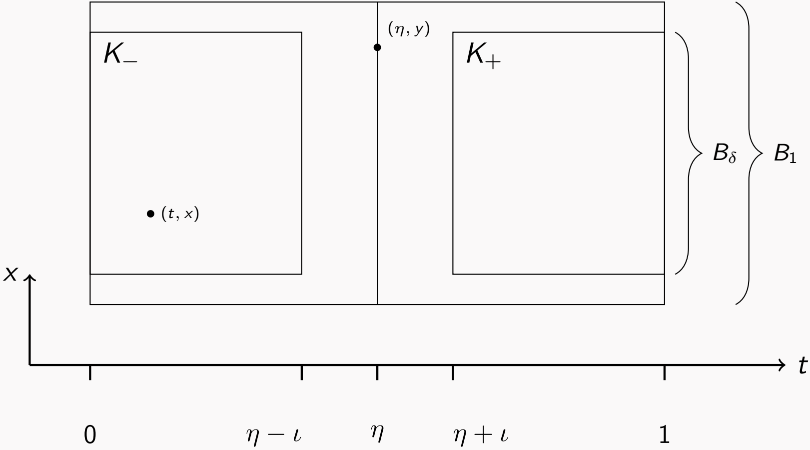
For $g = \log u$ we have

$$g(t,x) - c(u) = \frac{1}{c_\varphi} \int_B (g(t,x) - g(\eta,y))\varphi^2(y)\mathrm{d}y$$
$$= -\frac{1}{c_\varphi} \int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r))\mathrm{d}r \varphi^2(y)\mathrm{d}y$$

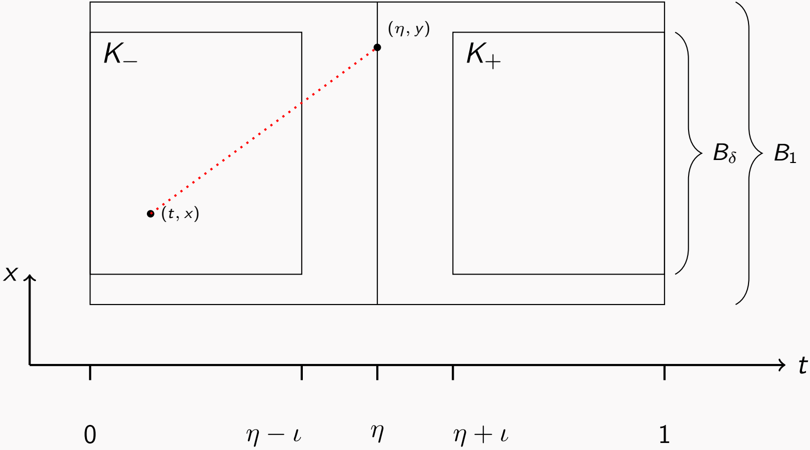
with $\gamma\colon [0,1] \to \mathbb{R} \times \mathbb{R}^n$ with $\gamma(0) = (t,x)$ and $\gamma(1) = (\eta, y)$.

What is a good choice for the trajectory γ?

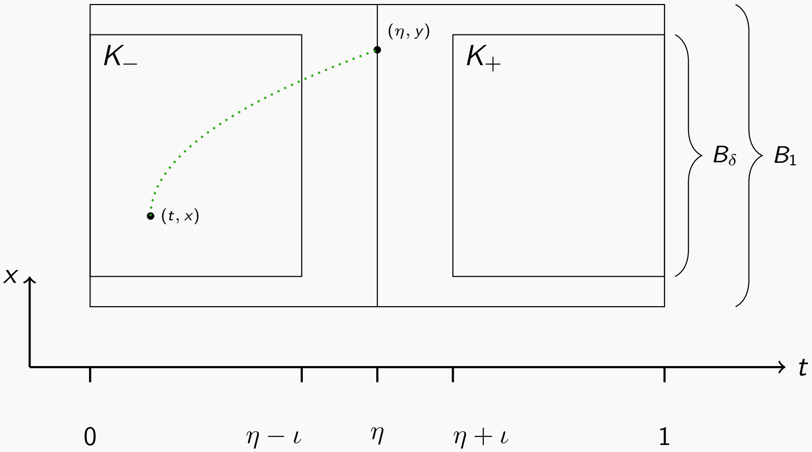
Trajectories



Trajectories



Parabolic trajectories



Proof using parabolic trajectories

For $g = \log u$ we have

$$g(t,x) - c(u) = \frac{1}{c_\varphi} \int_B (g(t,x) - g(\eta,y))\varphi^2(y)\mathrm{d}y$$

$$= -\frac{1}{c_\varphi} \int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r))\mathrm{d}r\varphi^2(y)\mathrm{d}y$$

Parabolic trajectory: $\gamma(r) = (t + r^2(\eta - t), x + r(y - x))$

Proof using parabolic trajectories

For $g = \log u$ we have

$$g(t,x) - c(u) = \frac{1}{c_\varphi}\int_B (g(t,x) - g(\eta,y))\varphi^2(y)\mathrm{d}y$$
$$= -\frac{1}{c_\varphi}\int_B\int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r))\mathrm{d}r\varphi^2(y)\mathrm{d}y$$
$$= -\frac{1}{c_\varphi}\int_0^1\int_B \big(2(\eta - t)r[\partial_t g](\gamma(r)) + (y-x)\cdot[\nabla g](\gamma(r))\big)\varphi^2(y)\mathrm{d}y\mathrm{d}r,$$

Parabolic trajectory: $\gamma(r) = (t + r^2(\eta - t), x + r(y-x))$

Proof using parabolic trajectories

For $g = \log u$ we have

$$g(t,x) - c(u) = \frac{1}{c_\varphi}\int_B (g(t,x) - g(\eta,y))\varphi^2(y)\mathrm{d}y$$

$$= -\frac{1}{c_\varphi}\int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r))\mathrm{d}r\varphi^2(y)\mathrm{d}y$$

$$= -\frac{1}{c_\varphi}\int_0^1 \int_B \big(2(\eta - t)r[\partial_t g](\gamma(r)) + (y-x)\cdot[\nabla g](\gamma(r))\big)\varphi^2(y)\mathrm{d}y\mathrm{d}r$$

$$\le \frac{1}{c_\varphi}\int_0^1 \int_B \Big(-2(\eta - t)r[\nabla\cdot(A\nabla g)](\gamma(r)) - 2(\eta-t)r[\langle A\nabla g, \nabla g\rangle](\gamma(r))$$
$$-(y-x)\cdot[\nabla g](\gamma(r))\Big)\varphi^2(y)\mathrm{d}y\mathrm{d}r.$$

Parabolic trajectory: $\gamma(r) = (t + r^2(\eta - t), x + r(y-x))$
Idea: use quadratic gradient term to absorb all gradients

Proof using parabolic trajectories

For $g = \log u$ we have

$$\begin{aligned} g(t,x) - c(u) &= \frac{1}{c_\varphi}\int_B (g(t,x) - g(\eta,y))\varphi^2(y)\mathrm{d}y \\ &= -\frac{1}{c_\varphi}\int_B\int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r))\mathrm{d}r\varphi^2(y)\mathrm{d}y \\ &= -\frac{1}{c_\varphi}\int_0^1\int_B \big(2(\eta - t)r[\partial_t g](\gamma(r)) + (y-x)\cdot[\nabla g](\gamma(r))\big)\varphi^2(y)\mathrm{d}y\mathrm{d}r \\ &\le \frac{1}{c_\varphi}\int_0^1\int_B \Big(-2(\eta - t)r[\nabla\cdot(A\nabla g)](\gamma(r)) - 2(\eta - t)r[\langle A\nabla g, \nabla g\rangle](\gamma(r)) \\ &\qquad -(y-x)\cdot[\nabla g](\gamma(r))\Big)\varphi^2(y)\mathrm{d}y\mathrm{d}r. \end{aligned}$$

Parabolic trajectory: $\gamma(r) = (t + r^2(\eta - t), x + r(y-x))$
Idea: use quadratic gradient term to absorb all gradients

Partial integration

Substitute $\tilde{y} = \Phi(y) = \Phi_{r,t,x,\eta}(y) := \gamma_x(r)$, hence

$$\begin{aligned}
&\int_0^1 \int_B -r[\nabla \cdot (A\nabla g)](\gamma(r))\varphi^2(y)\mathrm{d}y\mathrm{d}r \\
&= -\int_0^1 \int_{\Phi(B)} [\nabla \cdot (A\nabla g)](\gamma_t(r), \tilde{y})\varphi^2 \left(\frac{1}{r}\tilde{y} + \left(1 - \frac{1}{r}\right)x\right) r^{-d+1}\mathrm{d}\tilde{y}\mathrm{d}r \\
&= 2\int_0^1 \int_{\Phi(B)} (A\nabla g)(\gamma_t(r), \tilde{y}) \cdot [\nabla\varphi] \left(\frac{1}{r}\tilde{y} + \left(1 - \frac{1}{r}\right)x\right) \\
&\qquad \varphi\left(\frac{1}{r}\tilde{y} + \left(1 - \frac{1}{r}\right)x\right) r^{-d}\mathrm{d}\tilde{y}\mathrm{d}r \\
&= 2\int_0^1 \int_B (A\nabla g)(\gamma(r)) \cdot [\nabla\varphi](y)\varphi(y)\mathrm{d}y\mathrm{d}r \\
&\leq \frac{4\sqrt{\Lambda}}{1-\delta} \int_0^1 \int_B |\nabla g|_A (\gamma(r))\varphi(y)\mathrm{d}y\mathrm{d}r,
\end{aligned}$$

Here: $|\xi|_A^2 := \langle A(t,x)\xi, \xi\rangle$.

Distributing the good term

$$
\begin{aligned}
&g(t,x) - c(u) \\
&\le \frac{1}{c_\varphi} \int_0^1 \int_B \Big(-2(\eta - t) r [\nabla \cdot (A \nabla g)](\gamma(r)) - (\eta - t) r \left| \nabla g \right|_A^2 (\gamma(r)) \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \varphi^2(y) \mathrm{d}y \mathrm{d}r \\
&\quad + \frac{1}{c_\varphi} \int_0^1 \int_B \Big(-(y - x) \cdot [\nabla g](\gamma(r)) - (\eta - t) r \left| \nabla g \right|_A^2 (\gamma(r)) \Big) \varphi^2(y) \mathrm{d}y \mathrm{d}r \\
&\le \frac{\eta - t}{c_\varphi} \int_0^1 \int_B \left(\frac{8\sqrt{\Lambda}}{1 - \delta} \left| \nabla g \right|_A (\gamma(r)) \varphi(y) - r \left| \nabla g \right|_A^2 (\gamma(r)) \varphi^2(y) \right) \mathrm{d}y \mathrm{d}r \\
&\quad + \frac{1}{c_\varphi} \int_0^1 \int_B \left(\frac{2}{\sqrt{\lambda}} \left| \nabla g \right|_A (\gamma(r)) \varphi(y) - r(\eta - t) \left| \nabla g \right|_A^2 (\gamma(r)) \varphi^2(y) \right) \mathrm{d}y \mathrm{d}r.
\end{aligned}
$$

Integrating on K_-

$$\int_0^{\eta-\iota}\int_B (g(t,x)-c(u))_+ \mathrm{d}x\mathrm{d}t \leq$$

$$\frac{1}{c_\varphi}\int_0^{\eta-\iota}(\eta-t)\int_B\int_B\int_0^1\left(\frac{8\sqrt{\Lambda}}{1-\delta}\left|\nabla g\right|_A(\gamma(r))\varphi(y)-r\left|\nabla g\right|_A^2(\gamma(r))\varphi^2(y)\right)_+\mathrm{d}r\mathrm{d}y\mathrm{d}x\mathrm{d}t$$

$$+\frac{1}{c_\varphi}\int_0^{\eta-\iota}(\eta-t)\int_B\int_B\int_0^1\left(\frac{4\lambda^{-1/2}}{(\eta-t)}\left|\nabla g\right|_A(\gamma(r))\varphi(y)-r\left|\nabla g\right|_A^2(\gamma(r))\varphi^2(y)\right)_+\mathrm{d}r\mathrm{d}y\mathrm{d}x\mathrm{d}t$$

Let $M > 0$. Then

$$\begin{aligned}
&\int_0^{\eta-\iota} \int_B \int_B \int_0^1 \Big(M \left|\nabla g\right|_A (\gamma(r))\varphi(y) - r \left|\nabla g\right|_A^2 (\gamma(r))\varphi^2(y) \Big)_+ \mathrm{d}r\mathrm{d}y\mathrm{d}x\mathrm{d}t \\
&= \int_0^{\eta-\iota} \int_B \int_B \int_0^{1/2} \Big(M \left|\nabla g\right|_A (\gamma(r))\varphi(y) - r \left|\nabla g\right|_A^2 (\gamma(r))\varphi^2(y) \Big)_+ \mathrm{d}r\mathrm{d}y\mathrm{d}x\mathrm{d}t \\
&+ \int_0^{\eta-\iota} \int_B \int_B \int_{1/2}^1 \Big(M \left|\nabla g\right|_A (\gamma(r))\varphi(y) - r \left|\nabla g\right|_A^2 (\gamma(r))\varphi^2(y) \Big)_+ \mathrm{d}r\mathrm{d}y\mathrm{d}x\mathrm{d}ts \\
&=: I_1 + I_2 \\
&\le I_1 + C
\end{aligned}$$

for $C > 0$ by Cauchy-Schwarz inequality.

I_1

Substitute $\tilde{x} = \Psi_{r,t,\eta,y}(x) := \gamma_x(r)$ and $\tilde{t} = t + r^2(\eta - t)$. Abbreviate $p(\tilde{t}, \tilde{x}, \eta, y) = |\nabla g|_{A(\tilde{t},\tilde{x})}(\tilde{t}, \tilde{x})\varphi(y)$ and

$$
\begin{aligned}
I_1 &= \int_B \int_0^{1/2} \int_{r^2\eta}^{\eta+(r^2-1)\iota} \int_{\Psi(B)} \Big(M\, |\nabla g|_A\, (\tilde{t}, \tilde{x})\varphi(y) - r\, |\nabla g|_A^2\, (\tilde{t}, \tilde{x})\varphi^2(y) \Big)_+ \\
&\qquad \cdot (1-r)^{-d}(1-r^2)^{-1} \mathrm{d}\tilde{x}\mathrm{d}\tilde{t}\mathrm{d}r\mathrm{d}y \\
&\le C \int_B \int_0^{1/2} \int_0^{\eta} \int_B \left(Mp - rp^2\right)_+ \mathrm{d}\tilde{x}\mathrm{d}\tilde{t}\mathrm{d}r\mathrm{d}y \\
&= C \int_B \int_0^{\eta} \int_B \int_0^{1/2} \left(Mp - rp^2\right)_+ \mathrm{d}r\mathrm{d}\tilde{x}\mathrm{d}\tilde{t}\mathrm{d}y
\end{aligned}
$$

as $\Psi(B) \subset B$ for some $C = C(d)$. Considering the inner integral with $m = \min\{1/2, M/p\}$

$$
\int_0^{1/2} \left(Mp - rp^2\right)_+ \mathrm{d}r = mMp - \frac{m^2}{2}p^2 \le \frac{M^2}{\sqrt{2}}
$$

for all $p > 0$.

Proof of the weak L^1-estimate mod.

With

$$c(u) = \frac{1}{c_\varphi} \int_B [\log u](\eta, y)\varphi^2(y)\mathrm{d}y$$

we obtain

$$\int_0^{\eta-\iota} \int_B ([\log u](t,x) - c(u))_+ \mathrm{d}x\mathrm{d}t \leq C$$

for some $C > 0$.

- $\gamma(r) = (t + r^k(\eta - t), x + r^j(y - x))$ with $j, k > 0$ works if $k = 2j$
- reminiscent of the proof via Li-Yau '86 inequality $-\Delta \log u \leq \frac{n}{2t}$
- formal calculations
- first proof which does not follow the strategy of Moser

Kinetic equations

Here: $x, v \in \mathbb{R}^n$, $t \in [0, T]$, $u = u(t, x, v)$ particle density

$$\partial_t u + v \cdot \nabla_x u = \nabla_v \cdot (A(t, x, v) \nabla_v u)$$

Kolmogorov equation

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with $A\colon [0, T] \times \mathbb{R}^{2n} \to \mathbb{R}^{n\times n}$ measurable, elliptic and bounded.

- Kolmogorov equation with rough coefficients
- linearised version of the Landau equation
- Kolmogorov constructed fundamental solution in '34

Kolmogorov equation

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- Kolmogorov equation with rough coefficients
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Kolmogorov equation

Consider $A = \mathrm{Id}$.

Hörmander operator (type B) - hypoelliptic

$$(\partial_t + v \cdot \nabla_x)u = \sum_{i=1}^{n} \partial_{v_i}^2 u + f$$

Kolmogorov equation

Consider $A = \mathrm{Id}$.

Hörmander operator (type B) - hypoelliptic

$$X_0 u = \sum_{i=1}^{n} X_i^2 u + f$$

where $X_0 = \partial_t + v \cdot \nabla_x$ and $X_i = \partial_{v_i}$.

$$[\partial_{v_i}, \partial_t + v \cdot \nabla_x]u = \partial_{v_i}(\partial_t + v \cdot \nabla_x)u - (\partial_t + v \cdot \nabla_x)\,\partial_{v_i} u = \partial_{x_i} u$$

Kolmogorov equation

Consider $A = \mathrm{Id}$.

Hörmander operator (type B) - hypoelliptic

$$X_0 u = \sum_{i=1}^{n} X_i^2 u + f$$

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$$[\partial_{v_i}, \partial_t + v \cdot \nabla_x]u = \partial_{v_i}(\partial_t + v \cdot \nabla_x)u - (\partial_t + v \cdot \nabla_x)\, \partial_{v_i} u = \partial_{x_i} u$$

Theorem (Hörmander '67): If $f \in C^\infty$, then $u \in C^\infty$

Kinetic geometry

Consider $A = \mathrm{Id}$.

$$\partial_t u + v \cdot \nabla_x u = \Delta_v u + f$$

Scaling invariance:

$\lambda \mapsto (\lambda^2 t, \lambda^3 x, \lambda v)$

Translation invariance:

$(t_0, x_0, v_0) \mapsto (t - t_0, x - x_0 - (t - t_0)v, v - v_0)$

Kinetic cylinders:

$Q_r(t_0, x_0, v_0)$

$= \{-r^2 < t - t_0 \leq 0,\ |x - x_0 - (t - t_0)v_0| < r^3,\ |v - v_0| < r^3\}$

Kinetic De Giorgi-Nash-Moser theory

We want a priori estimates for weak solutions of

$$\partial_t u + v \cdot \nabla_x u = \nabla_v \cdot (A(t,x,v)\nabla_v u)$$

where $A = A(t,x,v)$ is elliptic, bounded and measurable.

- Local boundedness by Pascucci & Polidoro '04
- A priori Hölder estimate by Wang & Zhang '09
- Harnack inequality by Golse, Imbert, Mouhot & Vasseur '19
- existence of weak solutions by Litsgård and Nyström '21
- many more recent works by Anceschi, Citti, Dietert, Guerand, Hirsch, Loher, Manfredini, Rebucci, Sire, Zhu

Can Moser's method be applied in the kinetic setting?

Kinetic Trajectories

Find $\gamma\colon [0,1] \to \mathbb{R}^{1+2n}$ satisfying

- $\gamma(0) = (t, x, v)$, $\gamma(1) = (\eta, y, w)$,
- γ moves along $\partial_t + v \cdot \nabla_x$ and ∇_v, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) = \dot{\gamma}_t(r)[\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot{\gamma}_v \cdot [\nabla_v g](\gamma(r))$$

for smooth g.

Kinetic Trajectories

Find $\gamma\colon [0,1] \to \mathbb{R}^{1+2n}$ satisfying

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for smooth g.

Studied by Carathéodory '09, Chow '40, Pascucci & Polidoro '04, ...

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for smooth g.

Studied by Carathéodory '09, Chow '40, Pascucci & Polidoro '04, ...

For the trajectorial proof we need regular trajectories, e.g.

$$\left|\partial_w \Phi^{-1}_{r,t,x,v}(y, w)\right| \lesssim r^{-1}$$

where $Phi_{r,t,x,v}(y, w) = (\gamma_2(r), \ldots, \gamma_{2n+1}(r))$.

Kinetic Trajectories

Find $\gamma\colon [0,1] \to \mathbb{R}^{1+2n}$ satisfying

– $\gamma(0) = (t, x, v)$, $\gamma(1) = (\eta, y, w)$,

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$$\frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) = \dot{\gamma}_t(r)[\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot{\gamma}_v \cdot [\nabla_v g](\gamma(r))$$

for smooth g.

Studied by Carathéodory '09, Chow '40, Pascucci & Polidoro '04, ...

We can construct kinetic trajectories with

$$\left|\partial_w \Phi^{-1}_{r,t,x,v}(y, w)\right| \lesssim r^{-1-\varepsilon}$$

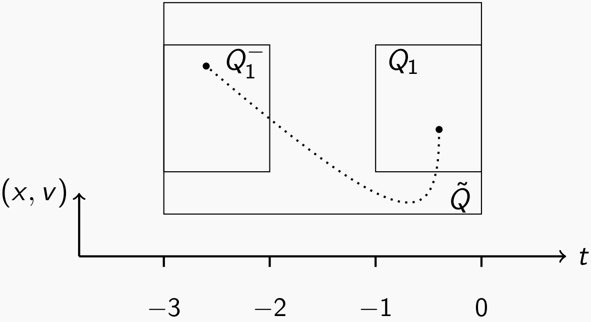
where $Phi_{r,t,x,v}(y, w) = (\gamma_2(r), \ldots, \gamma_{2n+1}(r))$.

Kinetic Poincaré inequality

Theorem (Guerand & Mouhot '22, N. & Zacher '22):

Let $A \in L^\infty(\tilde{Q}; \mathbb{R}^{n\times n})$ and φ^2 be supported in Q_1^-. Then there exists a constant $C = C(\|A\|_\infty, n, \varphi) > 0$ such that for all subsolutions $u \geq 0$ to (1) in $\tilde{Q}$ we have

$$\left\| (u - \langle u\varphi^2 \rangle_{Q_1^-})_+ \right\|_{L^1(Q_1)} \leq C \|\nabla_v u\|_{L^1(\tilde{Q})}.$$



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lukasniebel.github.io