Analytic aspects of kinetic partial differential equations

Promotionskolloquium von

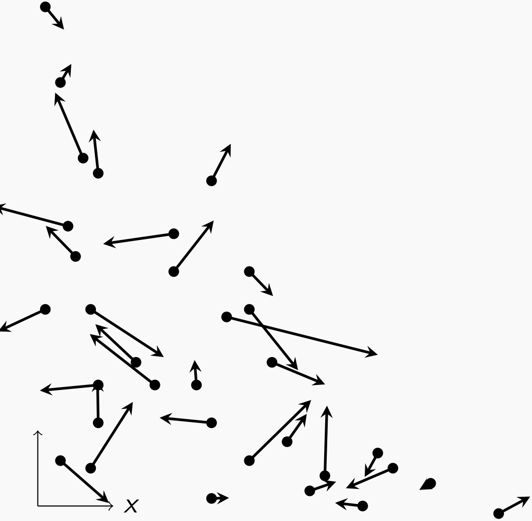
Lukas Niebel
Institut für Angewandte Analysis, Universität Ulm

14 Uhr am 5. Juni 2023

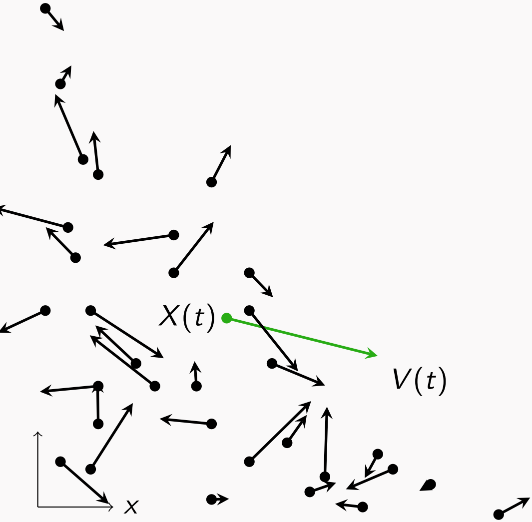
Analytic aspects

of kinetic partial differential equations

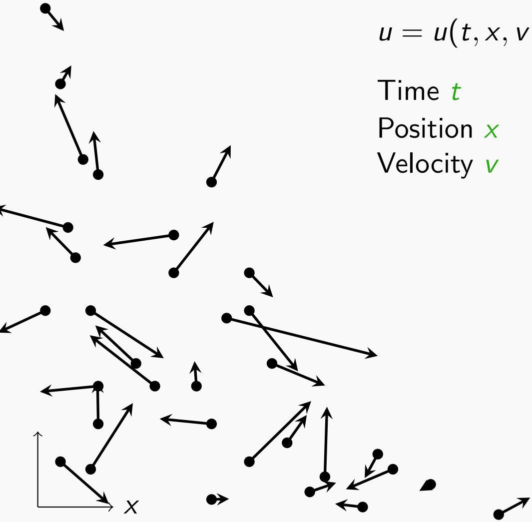
Particle physics



Particle physics



Particle physics



Particle distribution function

$$u = u(t, x, v)\colon [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

Time t

Position x

Velocity v

Free transport

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = f \\ u(0) = g \end{cases}$$

Today:

- $f = f(t, x, v)$ is a given source term
- $g = g(x, v)$ is the initial distribution.

Boltzmann equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = Q_B(u,u) + f \\ u(0) = g \end{cases}$$

with

$$Q_B(u,u) = \int_{\mathbb{R}^n} \int_{S^{n-1}} \big(u(v_*')u(v') - u(v_*)u(v) \big) B(v - v_*, \sigma) \, \mathrm{d}v_* \, \mathrm{d}\sigma,$$

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v_*' = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma$$

and a function $B \colon \mathbb{R}^n \times S^{n-1} \to [0, \infty)$.

Landau equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \bar{a}(u) \colon \nabla_v^2 u + \bar{c}(u) u + f \\ u(0) = g \end{cases}$$

with

$$\bar{a}(u) = a_{\gamma,n} \int_{\mathbb{R}^n} \Big(\mathrm{I}_n - \frac{w}{|w|} \otimes \frac{w}{|w|} \Big) |w|^{\gamma+2} u(t,x,v-w) \, \mathrm{d}w$$

and

$$\bar{c}(u) = c_{\gamma,n} \int_{\mathbb{R}^n} |w|^\gamma u(t,x,v-w) \, \mathrm{d}w.$$

Landau equation (simplified)

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = a \colon \nabla_v^2 u + c\, u + f \\ u(0) = g \end{cases}$$

with

$$a = a(t, x, v) \colon [0, T] \times \mathbb{R}^{2n} \to \mathbb{R}^{n \times n}$$

and

$$c = c(t, x, v) \colon [0, T] \times \mathbb{R}^{2n} \to \mathbb{R}.$$

Landau equation (simplified)

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = a \colon \nabla_v^2 u + c\, u + f \\ u(0) = g \end{cases}$$

with

$$a = \mathrm{I}_n$$

and

$$c = 0.$$

Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

– Kolmogorov 1934

– degenerate but hypoelliptic

(Fractional) Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\beta/2} u + f \\ u(0) = g \end{cases}$$

for $\beta \in (0, 2]$.

Analytic aspects

of kinetic partial differential equations

(Fractional) Kolmogorov equation

Kinetic maximal L^p-regularity

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\beta/2} u + f \\ u(0) = g \end{cases}$$

for $\beta \in (0, 2]$.

(Fractional) Kolmogorov equation

Kinetic maximal L^p-regularity

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\beta/2} u + f \\ u(0) = g \end{cases}$$

for $\beta \in (0, 2]$.

Goal: Determine function spaces X for f, X_γ for g and Z for u such that there exists a unique solution $u \in Z$ of the (fractional) Kolmogorov equation if and only if $f \in X$ and $g \in X_\gamma$.

Kinetic maximal L^p-regularity

Definition (N. & Zacher '22) *simplified*:

We say that $A \colon D(A) \subset L^p(\mathbb{R}^{2n}) \to L^p(\mathbb{R}^{2n})$ admits kinetic maximal L^p-regularity if for all $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$ there exists a unique distributional solution

$$u \in Z = \{w \, : \, w, \partial_t w + v \cdot \nabla_x w, Aw \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

of the Cauchy problem

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = Au + f \\ u(0) = 0 \end{cases}$$

with

$$\|u\|_p + \|\partial_t u + v \cdot \nabla_x u\|_p + \|Au\|_p \leq C \|f\|_p$$

for some constant $C = C(T,p) > 0$.

Kinetic maximal L^p-regularity

Corollary (N. & Zacher '22):

If A admits kinetic maximal L^p-regularity, then the Cauchy problem

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = Au + f \\ u(0) = g \end{cases}$$

admits a unique solution

$$u \in Z = \{w : \ w, \partial_t w + v \cdot \nabla_x w, Aw \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

if and only if

(i) $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$

(ii) $g \in X_\gamma = \{g : \exists u \in Z \text{ with } u(0) = g\}$ with $\|g\|_{X_\gamma} = \inf_{\substack{u \in Z \\ u(0)=g}} \|u\|_Z$.

Moreover, $u \in C([0,T]; X_\gamma)$.

Kolmogorov equation

Theorem (Folland et al. '74, Bramanti et al. '10, Dong et al. '22):

For all $p \in (1, \infty)$, the operator $\Delta_v : H_v^{2,p}(\mathbb{R}^{2n}) \to L^p(\mathbb{R}^{2n})$ admits kinetic maximal $L^p(L^p)$-regularity for all $p \in (1, \infty)$.

Proof: Singular integral theory on homogeneous groups.

Kolmogorov equation

Theorem (N. & Zacher '22):

For the Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

there exists a unique solution

$$u \in Z = \{w : \; w, \, \partial_t w + v \cdot \nabla_x w, \, \Delta_v w \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

if and only if

(i) $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$

(ii) $g \in X_\gamma$.

Moreover, $u \in C([0,T]; X_\gamma)$.

Kinetic trace

Recall:

$$X_\gamma = \{g \colon \exists u \in Z \text{ with } u(0) = g\} \text{ with } \|g\|_{X_\gamma} = \inf_{\substack{u \in Z \\ u(0) = g}} \|u\|_Z.$$

Kinetic trace

Recall:

$$X_\gamma = \{g\colon \exists u \in Z \text{ with } u(0) = g\} \text{ with } \|g\|_{X_\gamma} = \inf_{\substack{u\in Z\\ u(0)=g}} \|u\|_Z.$$

For the homogeneous problem $u = u(t, v)$

$$\begin{cases} \partial_t u = \Delta_v u + f \\ u(0) = g \end{cases} \qquad \text{(heat equation)}$$

we have $X_\gamma = B^{2(1-1/p)}_{pp,v}(\mathbb{R}^n)$.

Kinetic trace

Recall:

$X_\gamma = \{g \colon \exists u \in Z \text{ with } u(0) = g\}$ with $\|g\|_{X_\gamma} = \inf\limits_{\substack{u \in Z \\ u(0) = g}} \|u\|_Z$.

For the homogeneous problem $u = u(t, v)$

$$\begin{cases} \partial_t u = \Delta_v u + f \\ u(0) = g \end{cases} \qquad \text{(heat equation)}$$

we have $X_\gamma = B^{2(1-1/p)}_{pp,v}(\mathbb{R}^n)$.

Kinetic regularisation (Bouchut '02): $Z \hookrightarrow L^p((0, T); H_x^{\frac{2}{3}}(\mathbb{R}^{2n}))$.

Kinetic trace

Recall:

$$X_\gamma = \{g \colon \exists u \in Z \text{ with } u(0) = g\} \text{ with } \|g\|_{X_\gamma} = \inf_{\substack{u\in Z \\ u(0)=g}} \|u\|_Z.$$

Theorem (N. & Zacher '22):

Let $p \in (1, \infty)$ and X_γ the trace space to

$$Z = \{u \, : \; u,\, \partial_t u + v \cdot \nabla_x u,\, \Delta_v u \in L^p((0, T); L^p(\mathbb{R}^{2n}))\}.$$

Then

$$X_\gamma \cong B_{pp,x}^{\frac{2}{3}(1-\frac{1}{p})}(\mathbb{R}^{2n}) \cap B_{pp,v}^{2(1-\frac{1}{p})}(\mathbb{R}^{2n}).$$

Proof: Littlwood-Paley decomposition, Mikhlin multiplier theorem, and the fundamental solution.

Kolmogorov equation

Theorem (N. & Zacher '22):

For the Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

there exists a unique solution

$$u \in Z = \{w : \; w, \, \partial_t w + v \cdot \nabla_x w, \, \Delta_v w \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

if and only if

(i) $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$

(ii) $g \in X_\gamma = B_{pp,x}^{\frac{2}{3}(1-\frac{1}{p})}(\mathbb{R}^{2n}) \cap B_{pp,v}^{2(1-\frac{1}{p})}(\mathbb{R}^{2n})$.

Moreover, $u \in C([0,T]; X_\gamma)$.

Fractional Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\beta/2} u + f \\ u(0) = g. \end{cases}$$

Theorem (Chen & Zhang '18; Huang, Menozzi & Priola '19):

For $\beta \in (0,2)$ the operator $-(-\Delta_v)^{\beta/2} \colon H_v^{\beta,p}(\mathbb{R}^{2n}) \to L^p(\mathbb{R}^{2n})$ admits kinetic maximal $L^p(L^p)$-regularity for all $p \in (1,\infty)$.

Theorem (N. & Zacher '22):

$$X_\gamma \cong B_{pp,x}^{\frac{\beta}{\beta+1}(1-\frac{1}{p})}(\mathbb{R}^{2n}) \cap B_{pp,v}^{\beta(1-\frac{1}{p})}(\mathbb{R}^{2n})$$

Temporal weights

Replace $L^p((0,T);X)$ with

$$L^p_\mu((0,T);X) = \{u\colon\, t^{1-\mu} u \in L^p((0,T);X)\}$$

with $\mu \in (1/p, 1]$ (Muckenhoupt weight, Prüss & Simonett '04).

Temporal weights

Replace $L^p((0,T);X)$ with

$$L^p_\mu((0,T);X) = \{u\colon t^{1-\mu}u \in L^p((0,T);X)\}$$

with $\mu \in (1/p, 1]$ (Muckenhoupt weight, Prüss & Simonett '04).

Advantages:

- Theorem (N. & Zacher '22):
  Kinetic maximal L^p_μ-regularity is independent of $\mu \in (1/p, 1]$.
- For (fractional) Kolmogorov equation:

$$X_{\gamma,\mu} \cong B^{\frac{\beta}{\beta+1}(\mu-\frac{1}{p})}_{pp,x}(\mathbb{R}^{2n}) \cap B^{\beta(\mu-\frac{1}{p})}_{pp,v}(\mathbb{R}^{2n}).$$

- They allow to observe instantaneous regularisation.

Different base spaces

Theorem(s) (N. & Zacher '22,'23):

- Kinetic maximal $L^p(L^q)$-regularity for $-(-\Delta_v)^{\beta/2}$ with $p, q \in (1, \infty)$.

- Kinetic maximal $L^p(L^q_{j,k})$-regularity for Δ_v with $p, q \in (1, \infty)$ and $j, k \in \mathbb{R}$ where $L^q_{j,k}$ is weighted with $(1 + |v|)^j$ and $(1 + |x| + |v|)^k$.

- Kinetic maximal $L^p(X^{s,q}_\beta)$-regularity for $-(-\Delta_v)^{\beta/2}$
$$X^{s,q}_\beta = \left\{ f \in \mathcal{S}' : \ \left(1 + |\xi|^\beta + |k|^{\frac{\beta}{\beta+1}}\right)^s \mathcal{F}(f) \in L^q \right\}$$
with $p, q \in (1, \infty)$, $s \geq 0$ and $p \in (1, \infty)$, $q = 2$, $s \geq -1/2$.

Kolmogorov equation with variable coefficients

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = a(t, x, v) \colon \nabla_v^2 u + f \\ u(0) = g \end{cases}$$

Under which assumptions on the coefficient $a(t, x, v)$
do we obtain kinetic maximal L^p-regularity?

Kolmogorov equation with variable coefficients

Theorem (Bramanti et al. '13, N. & Zacher '22):

Let $a = a(t,x,v) \in L^\infty([0,T] \times \mathbb{R}^{2n}; \mathrm{Sym}(n))$ with $\lambda |\xi|^2 \leq \langle a(t,x,v)\xi, \xi\rangle$ for all (t,x,v) and $\xi \in \mathbb{R}^n$.

Kolmogorov equation with variable coefficients

Theorem (Bramanti et al. '13, N. & Zacher '22):

Let $a = a(t, x, v) \in L^\infty([0, T] \times \mathbb{R}^{2n}; \mathrm{Sym}(n))$ with $\lambda |\xi|^2 \leq \langle a(t, x, v)\xi, \xi\rangle$ for all (t, x, v) and $\xi \in \mathbb{R}^n$. Suppose

$\forall \varepsilon > 0\colon \exists \delta > 0$ such that $|t - s| + |x - y - (t - s)v| + |v - w| < \delta$ implies $|a(t, x, v) - a(s, y, w)| < \varepsilon \quad (\mathrm{BUC_{kin}})$

OR

$\forall \varepsilon > 0\colon \exists \delta > 0$ such that $|t - s| + |x - y| + |v - w| < \delta$ implies $|a(t, x, v) - a(s, y, w)| < \varepsilon \quad (\mathrm{BUC})$.

Kolmogorov equation with variable coefficients

Theorem (Bramanti et al. '13, N. & Zacher '22):

Let $a = a(t,x,v) \in L^\infty([0,T] \times \mathbb{R}^{2n}; \mathrm{Sym}(n))$ with $\lambda|\xi|^2 \leq \langle a(t,x,v)\xi,\xi\rangle$ for all (t,x,v) and $\xi \in \mathbb{R}^n$. Suppose

$\forall \varepsilon > 0$: $\exists \delta > 0$ such that $|t-s| + |x-y-(t-s)v| + |v-w| < \delta$ implies $|a(t,x,v) - a(s,y,w)| < \varepsilon \quad (\mathrm{BUC}_{\mathrm{kin}})$

OR

$\forall \varepsilon > 0$: $\exists \delta > 0$ such that $|t-s| + |x-y| + |v-w| < \delta$ implies $|a(t,x,v) - a(s,y,w)| < \varepsilon \quad (\mathrm{BUC})$.

Then the family of operators

$$A(t) = a(t,x,v) : \nabla_v^2 : H^{2,p}_{v,j,k}(\mathbb{R}^{2n}) \to L^p_{j,k}(\mathbb{R}^{2n})$$

admits kinetic maximal $L^p_\mu(L^q_{j,k})$-regularity.

Fractional Kolmogorov equation with variable density

Theorem (N. '22):

Let $\alpha \in (0,1)$ and $a = a(t,x,v,h) \in L^\infty([0,T] \times \mathbb{R}^{3n})$ symmetric in h with $0 < \lambda \le a \le \Lambda$ and

$$\sup \frac{|a(t,x,v,h) - a(s,y,w,h)|}{|t-s|^\alpha + |x-y-(t-s)v|^\alpha + |v-w|^\alpha} < \infty.$$

Then, the family of operators

$$A(t)u = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t,x,v+h) - u(t,x,v)}{|h|^{n+\beta}} a(t,x,v,h) \mathrm{d}h$$

admits kinetic maximal $L^p_\mu(L^p)$-regularity for all $p > \frac{n}{\alpha}$, $\mu \in (1/p, 1]$.

Same trace space as for $-(-\Delta_v)^{\beta/2}$.

Application to quasilinear equations

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = A(u)u + F(u) \\ u(0) = g \end{cases} \tag{1}$$

Application to quasilinear equations

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = A(u)u + F(u) \\ u(0) = g \end{cases} \tag{1}$$

Theorem (N. & Zacher '22):

Assume that

– $(A, F) \in C^{1-}_{\mathrm{loc}}(X_{\gamma,\mu}; \mathcal{B}(D, X) \times X)$

– $A(g)$ admits kinetic maximal $L^p_\mu(X)$-regularity.

Then there exists $T = T(g)$ and $\varepsilon = \varepsilon(g) > 0$ such that (1) admits a unique solution in Z for all $h \in \overline{B_\varepsilon(g)}^{X_{\gamma,\mu}}$.
Moreover, solutions depend continuously on the initial datum.

Here: $X = X^{s,q}_{\beta,j,k}$, $D \subset X$ and $Z = \mathcal{T}^p_\mu((0,T); X) \cap L^p_\mu((0,T); D)$.

A kinetic toy model

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = M(u)\Delta_v u \\ u(0) = g \end{cases} \tag{1}$$

with the local density $M(u)(t,x) = \int_{\mathbb{R}^n} u(t,x,v)\mathrm{d}v$.
(Villani '00, Liao et al. '18, Mouhot & Imbert '21, Anceschi & Zhu '21)

Theorem (N. & Zacher '23):

Let $j > n$, $\lambda > 0$, $p, q \in (1,\infty)$, $\mu \in (1/p, 1]$ with $\mu - 1/p > 2n/q$.
Then for every $g \in {}^{\mathrm{kin}}B^{\mu-1/p,2}_{qp,j}(\mathbb{R}^{2n})$ with $M(g) \geq \lambda$ there exists a time $T = T(g)$ such that (1) admits a unique solution

$$u \in \mathcal{T}^p_\mu((0,T); L^q_j(\mathbb{R}^{2n})) \cap L^p_\mu((0,T); H^{2,q}_{v,j}(\mathbb{R}^{2n})).$$

Note that: ${}^{\mathrm{kin}}B^{\mu-1/p,2}_{qp,j}(\mathbb{R}^{2n}) \hookrightarrow C_{0,j}(\mathbb{R}^{2n})$.

Kinetic De Giorgi-Nash-Moser theory

We want a priori estimates for weak solutions of

$$\partial_t u + v \cdot \nabla_x u = \nabla_v \cdot (A(t,x,v)\nabla_v u)$$

where $A = A(t,x,v)$ is elliptic, bounded and measurable.

- Local boundedness by Pascucci & Polidoro '04
- A priori Hölder estimate by Wang & Zhang '09
- Harnack inequality by Golse, Imbert, Mouhot & Vasseur '19
- many more recent works by Anceschi, Citti, Dietert, Guerand, Hirsch, Loher, Manfredini, Rebucci, Sire, Zhu

Can Moser's method be applied in the kinetic setting?

Parabolic Harnack inequality à la Moser

Let $\Omega \subset \mathbb{R}^n$ open and $T > 0$. Consider weak solutions $u = u(t,x) \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$ to

$$\partial_t u = \nabla \cdot (A \nabla u) \quad \text{in } (0,T) \times \Omega \tag{1}$$

where $\lambda \leq A = A(t,x) \leq \Lambda$ is measurable.

Parabolic Harnack inequality à la Moser

Let $\Omega \subset \mathbb{R}^n$ open and $T > 0$. Consider weak solutions $u = u(t,x) \in C([0,T];L^2(\Omega)) \cap L^2((0,T);H^1(\Omega))$ to

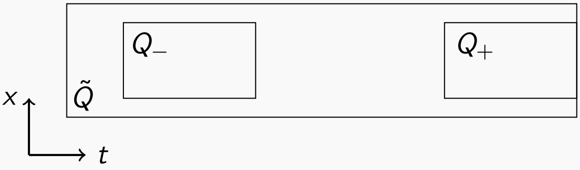
$$\partial_t u = \nabla \cdot (A \nabla u) \quad \text{in } (0,T)\times\Omega \tag{1}$$

where $\lambda \leq A = A(t,x) \leq \Lambda$ is measurable.

Theorem (Moser '64):

Let $\delta \in (0,1)$, $\tau > 0$. There exists $C = C(\delta, \lambda, \Lambda, n, \tau) > 0$ such that for any nonnegative weak solution u of (1) in $\tilde{Q}$ we have

$$\sup_{Q_-} u \leq C \inf_{Q_+} u.$$



Parabolic Harnack inequality à la Moser 1971

Three ingredients:

A: $L^p - L^\infty$ estimate for small $p \neq 0$

B: Weak L^1-Poincaré inequality for the logarithm of supersolutions

C: Lemma of Bombieri and Giusti

Parabolic Harnack inequality à la Moser 1971

Three ingredients:

A: $L^p - L^\infty$ estimate for small $p \neq 0$

B: Weak L^1-Poincaré inequality for the logarithm of supersolutions

C: Lemma of Bombieri and Giusti

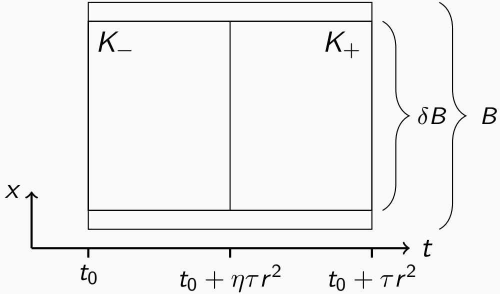
Weak L^1-Poincaré inequality for $\log u$

Theorem (Moser '64 & '71): $(1)\ \partial_t u = \nabla \cdot (A \nabla u)$

Let $\delta, \eta \in (0,1)$ and $\varepsilon, \tau > 0$. Then for any supersolution $u \geq \varepsilon > 0$ to (1) there exists constants $c = c(u)$ and $C = C(\delta, \eta, n, \tau) > 0$ s.t.

$$s\,|\{(t,x) \in K_-\colon\ \log u(t,x) - c(u) > s\}| \leq C(\tfrac{1}{\lambda} + \Lambda) r^2\,|B|\,,\ s > 0$$

$$s\,|\{(t,x) \in K_+\colon c(u) - \log u(t,x) > s\}| \leq C(\tfrac{1}{\lambda} + \Lambda) r^2\,|B|\,,\ s > 0.$$



Weak L^1-Poincaré inequality for $\log u$

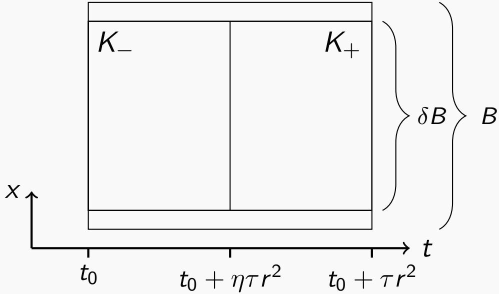
Theorem (Moser '64 & '71): $(1)\ \partial_t u = \nabla \cdot (A \nabla u)$

Let $\delta, \eta \in (0,1)$ and $\varepsilon, \tau > 0$. Then for any supersolution $u \geq \varepsilon > 0$ to (1) there exists constants $c = c(u)$ and $C = C(\delta, \eta, n, \tau) > 0$ s.t.

$$s \left|\{(t,x) \in K_- : \log u(t,x) - c(u) > s\}\right| \leq C(\tfrac{1}{\lambda} + \Lambda) r^2 |B|, \ s > 0$$

$$s \left|\{(t,x) \in K_+ : c(u) - \log u(t,x) > s\}\right| \leq C(\tfrac{1}{\lambda} + \Lambda) r^2 |B|, \ s > 0.$$

Note that: $\partial_t \log u \geq \nabla \cdot (A \nabla \log u) + \langle A \nabla \log u, \nabla \log u \rangle$.



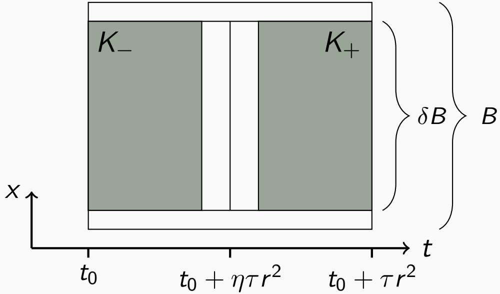
Weak L^1-Poincaré inequality for $\log u$ (weaker)

Theorem (N. & Zacher '22): (1) $\partial_t u = \nabla \cdot (A \nabla u)$

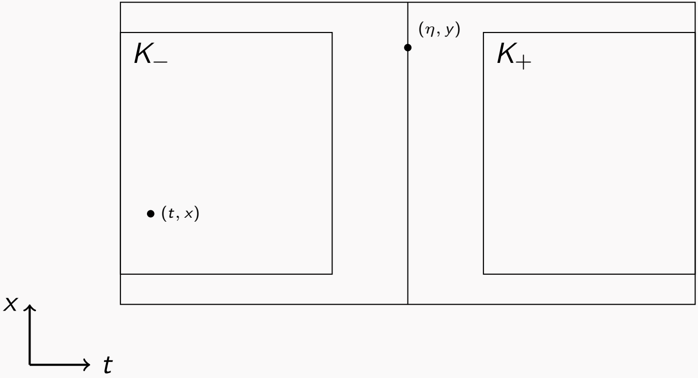
Let $\delta, \eta \in (0,1)$ and $\varepsilon, \tau > 0$. Then for any supersolution $u \geq \varepsilon > 0$ to (1) there exists constants $c = c(u)$ and $C = C(\delta, \eta, n, \tau) > 0$ s.t.

$$s \left|\{(t,x) \in K_- \colon \log u(t,x) - c(u) > s\}\right| \leq C(\tfrac{1}{\lambda} + \Lambda) r^2 \left|B\right|, \; s > 0$$

$$s \left|\{(t,x) \in K_+ \colon c(u) - \log u(t,x) > s\}\right| \leq C(\tfrac{1}{\lambda} + \Lambda) r^2 \left|B\right|, \; s > 0.$$

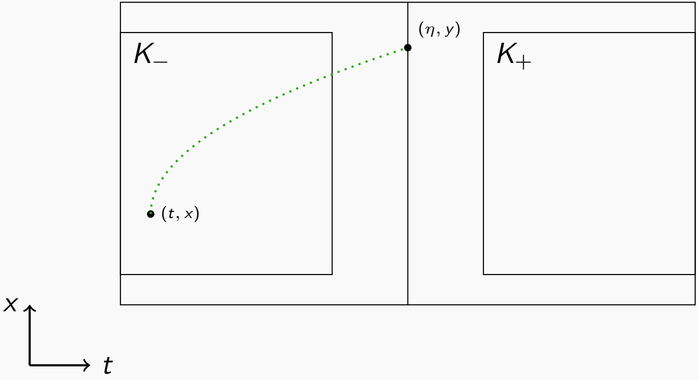


Parabolic trajectories



$$g(\eta, y) - g(t, x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r))\mathrm{d}r$$

Parabolic trajectories



$$g(\eta, y) - g(t, x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) \mathrm{d}r$$

$$\gamma(r) = (t + r^2(\eta - t), x + r(y - x))$$

Kinetic Trajectories

Find $\gamma\colon [0,1] \to \mathbb{R}^{1+2n}$ satisfying

– $\gamma(0) = (t, x, v)$, $\gamma(1) = (\eta, y, w)$,

– γ moves along $\partial_t + v \cdot \nabla_x$ and ∇_v, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) = \dot{\gamma}_t(r)[\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot{\gamma}_v \cdot [\nabla_v g](\gamma(r))$$

for smooth g.

Studied by Carathéodory '09, Chow '40, Pascucci & Polidoro '04, ...

Kinetic Trajectories

Find $\gamma\colon [0,1] \to \mathbb{R}^{1+2n}$ satisfying

– $\gamma(0) = (t, x, v)$, $\gamma(1) = (\eta, y, w)$,

– γ moves along $\partial_t + v \cdot \nabla_x$ and ∇_v, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) = \dot\gamma_t(r)[\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot\gamma_v \cdot [\nabla_v g](\gamma(r))$$

for smooth g.

Studied by Carathéodory '09, Chow '40, Pascucci & Polidoro '04, ...

For the trajectorial proof we need regular trajectories, e.g.

$$\left|\partial_w \Phi^{-1}_{r,t,x,v}(y,w)\right| \lesssim r^{-1}$$

where $Phi_{r,t,x,v}(y,w) = (\gamma_2(r), \dots, \gamma_{2n+1}(r))$.

Kinetic Trajectories

Find $\gamma\colon [0,1] \to \mathbb{R}^{1+2n}$ satisfying

– $\gamma(0) = (t, x, v)$, $\gamma(1) = (\eta, y, w)$,

– γ moves along $\partial_t + v \cdot \nabla_x$ and ∇_v, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) = \dot{\gamma}_t(r)[\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot{\gamma}_v \cdot [\nabla_v g](\gamma(r))$$

for smooth g.

Studied by Carathéodory '09, Chow '40, Pascucci & Polidoro '04, ...

We can construct kinetic trajectories with

$$\left|\partial_w \Phi^{-1}_{r,t,x,v}(y, w)\right| \lesssim r^{-1-\varepsilon}$$

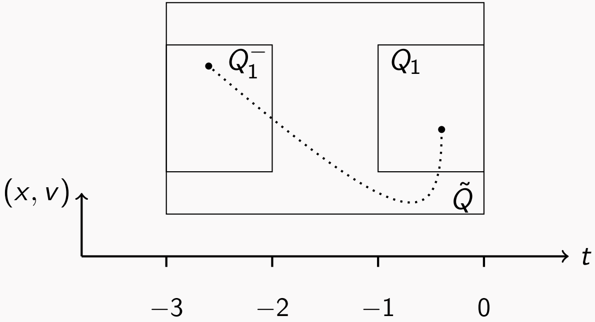
where $Phi_{r,t,x,v}(y, w) = (\gamma_2(r), \ldots, \gamma_{2n+1}(r))$.

Kinetic Poincaré inequality

Theorem (Guerand & Mouhot '22, N. & Zacher '22):

Let $A \in L^\infty(\tilde{Q}; \mathbb{R}^{n\times n})$ and φ^2 be supported in Q_1^-. Then there exists a constant $C = C(\|A\|_\infty, n, \varphi) > 0$ such that for all subsolutions $u \geq 0$ to (1) in $\tilde{Q}$ we have

$$\left\| (u - \langle u\varphi^2 \rangle_{Q_1^-})_+ \right\|_{L^1(Q_1)} \leq C \|\nabla_v u\|_{L^1(\tilde{Q})}.$$



Back to the kinetic toy model

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = M(u)\Delta_v u \\ u(0) = g \end{cases} \tag{1}$$

Theorem (N. & Zacher '23):

Assumptions as before. Let u be the solution to (1) with initial value $0 \leq g \in {}^{\mathrm{kin}}B^{\mu-1/p,2}_{qp,j}(\mathbb{R}^{2n})$ extended to $[0, T_{\max})$. If there exist $0 < M_0 < M_1$ such that

$$M_0 \leq M(u)(t,x) \leq M_1 \text{ for all } (t,x) \in [0, T_{\max}) \times \mathbb{R}^n$$

then $T_{\max} = \infty$.

Conditional global existence

References

- L. N., R. Zacher, *Kinetic maximal L^2-regularity for the (fractional) Kolmogorv equation*. Journal of Evolution Equations 21 (2021).
- L. N., R. Zacher, *Kinetic maximal L^p-regularity with temporal weights and application to quasilinear kinetic diffusion equations*. Journal of Differential Equations 307 (2022).
- L. N., Kinetic maximal $L^p_\mu(L^p)$-regularity for the fractional Kolmogorov equation with variable density. Nonlinear Analysis (2022).
- L. N and R. Zacher. *A trajectorial interpretation of Moser's proof of the Harnack inequality*. To appear in Annali della Scuola Normale Superiore di Pisa - Classe di Scienze (2023).
- L. N and R. Zacher. *On a kinetic Poincaré inequality and beyond*. Preprint. arXiv:2212.03199 (2022).