Kinetic maximal L^p-regularity

Lukas Niebel, joint work with Rico Zacher
Applied Mathematics - University Münster

Real analysis and geometry - CIRM, 12–16 June, 2023

Kinetic maximal L^p-regularity

Kinetic maximal L^p-regularity

Kinetic maximal L^p-regularity

Moving particles

Kinetic maximal L^p-regularity

Moving particles

Physics/Biology/Economics

Kinetic maximal L^p-regularity

Moving particles

Physics/Biology/Economics

First order differential operator $\partial_t + v \cdot \nabla_x$

Kinetic maximal L^p-regularity

Moving particles

Physics/Biology/Economics

First order differential operator $\partial_t + v \cdot \nabla_x$

Lebesgue spaces L^p with $p \in (1, \infty)$

Kinetic maximal L^p-regularity

Moving particles

Physics/Biology/Economics

First order differential operator $\partial_t + v \cdot \nabla_x$

optimal regularity estimates

Lebesgue spaces L^p with $p \in (1, \infty)$

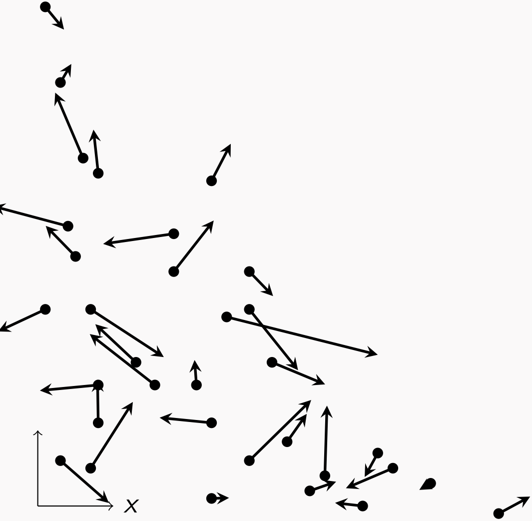
Kinetic maximal L^p-regularity

Moving particles

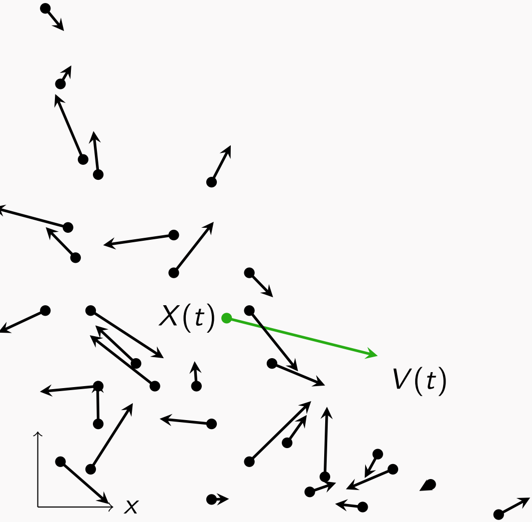
Physics/Biology/Economics

First order differential operator $\partial_t + v \cdot \nabla_x$

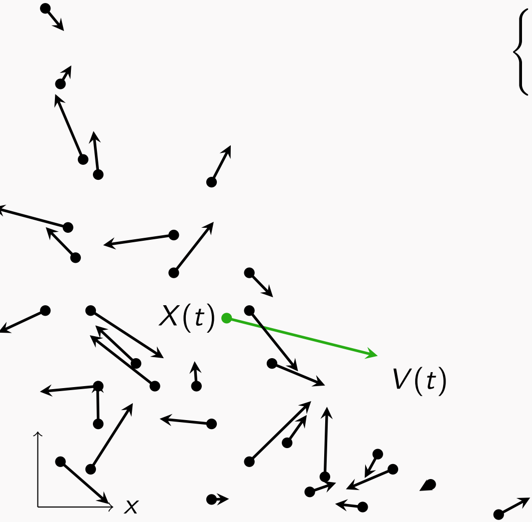
Particle physics



Particle physics



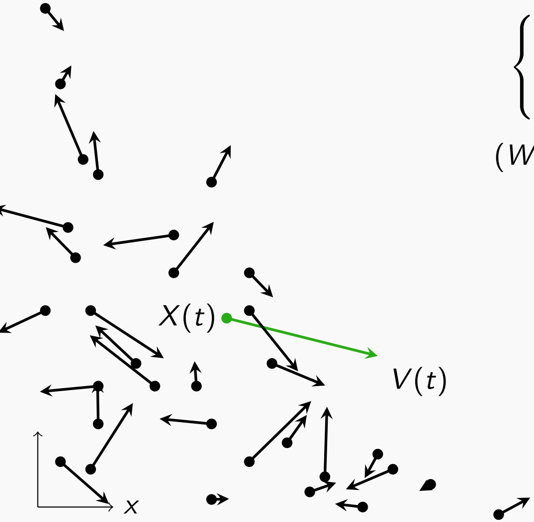
Particle physics



Free transport

$$\begin{cases} X(t) = tV_0 + X_0 \\ V(t) = V_0 \end{cases}$$

Particle physics

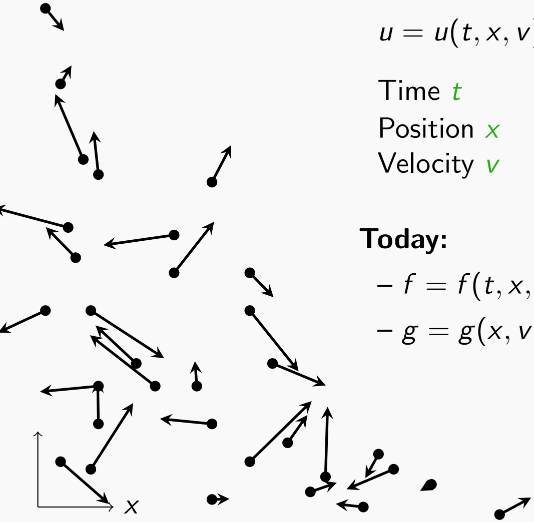


Simple collision model

$$\begin{cases} X(t) = \int_0^t V(s)\mathrm{d}s + X_0 \\ V(t) = W(t) + V_0 \end{cases}$$

$(W(t))_{t\geq 0}$ Wiener process

Particle physics



Particle distribution function

$$u = u(t, x, v) \colon [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

Time t

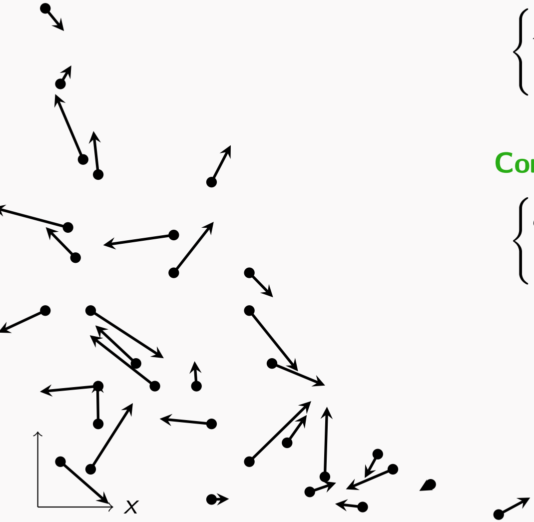
Position x

Velocity v

Today:

- $f = f(t, x, v)$ is a given source term
- $g = g(x, v)$ is the initial distribution.

Particle physics



Free transport

$$\begin{cases} X(t) = tV_0 + X_0 \\ V(t) = V_0 \end{cases}$$

Corresponding PDE

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = f \\ u(0) = g \end{cases}$$

Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

corresponds to the simple collision model.

Boltzmann equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = Q_B(u,u) + f \\ u(0) = g \end{cases}$$

with

$$Q_B(u,u) = \int_{\mathbb{R}^n} \int_{S^{n-1}} \big(u(v_*')u(v') - u(v_*)u(v)\big) B(v - v_*, \sigma)\,\mathrm{d}v_*\,\mathrm{d}\sigma,$$

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v_*' = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma$$

and a function $B \colon \mathbb{R}^n \times S^{n-1} \to [0,\infty)$.

Landau equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \bar{a}(u) \colon \nabla_v^2 u + \bar{c}(u) u + f \\ u(0) = g \end{cases}$$

with

$$\bar{a}(u) = a_{\gamma,n} \int_{\mathbb{R}^n} \Big(\mathrm{I}_n - \frac{w}{|w|} \otimes \frac{w}{|w|} \Big) |w|^{\gamma+2} u(t, x, v - w) \, \mathrm{d}w$$

and

$$\bar{c}(u) = c_{\gamma,n} \int_{\mathbb{R}^n} |w|^{\gamma} u(t, x, v - w) \, \mathrm{d}w.$$

Landau equation (simplified)

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = a \colon \nabla_v^2 u + c\, u + f \\ u(0) = g \end{cases}$$

with

$$a = a(t, x, v) \colon [0, T] \times \mathbb{R}^{2n} \to \mathbb{R}^{n \times n}$$

and

$$c = c(t, x, v) \colon [0, T] \times \mathbb{R}^{2n} \to \mathbb{R}.$$

Landau equation (simplified)

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = a \colon \nabla_v^2 u + c\, u + f \\ u(0) = g \end{cases}$$

with

$$a = \mathrm{I}_n$$

and

$$c = 0.$$

Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Kolmogorov equation

2nd order PDE, degenerate, unbounded lower order term
reminds of Ornstein-Uhlenbeck equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Kolmogorov equation

2nd order PDE, degenerate, unbounded lower order term
reminds of Ornstein-Uhlenbeck equation
Hörmander operator - hypoelliptic

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Kolmogorov equation

Hörmander operator - hypoelliptic

$$\begin{cases} X_0 u = \sum_{i=1}^{n} X_i^2 u + f \\ u(0) = g \end{cases}$$

where $X_0 = \partial_t + v \cdot \nabla_x$ and $X_i = \partial_{v_i}$.

$$[\partial_{v_i}, \partial_t + v \cdot \nabla_x]u = \partial_{v_i}(\partial_t + v \cdot \nabla_x)u - (\partial_t + v \cdot \nabla_x)\, \partial_{v_i} u = \partial_{x_i} u$$

Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Scaling: $\lambda \mapsto (\lambda^2 t, \lambda^3 x, \lambda v)$

Translation: $(t_0, x_0, v_0) \mapsto (t - t_0, x - x_0 - (t - t_0)v, v - v_0)$

Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Goal: Determine function spaces X for f, X_γ for g and Z for u such that there exists a unique solution $u \in Z$ of the Kolmogorov equation if and only if $f \in X$ and $g \in X_\gamma$.

Kolmogorov equation

Kinetic maximal regularity

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Goal: Determine function spaces X for f, X_γ for g and Z for u such that there exists a unique solution $u \in Z$ of the Kolmogorov equation if and only if $f \in X$ and $g \in X_\gamma$.

Kolmogorov equation

Kinetic maximal L^p-regularity

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Ansatz:

Kolmogorov equation

Kinetic maximal L^p-regularity

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Ansatz: $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$ with $p \in (1, \infty)$.

Kolmogorov equation

Kinetic maximal L^p-regularity

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Ansatz: $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$ with $p \in (1, \infty)$.

What is the solution space Z?
What is the trace space X_γ?

Kolmogorov equation

Kinetic maximal L^p-regularity

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Ansatz: $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$ with $p \in (1, \infty)$.

What is the solution space Z?
What is the trace space X_γ?
Divide and conquer

Kolmogorov equation

What is the solution space Z?

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = 0 \end{cases}$$

Parabolic ansatz

$$Z = \{u \,:\; u,\ \partial_t u,\ \Delta_v u - v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}.$$

Kolmogorov equation

What is the solution space Z?

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = 0 \end{cases}$$

Parabolic ansatz

$$\cancel{Z = \{u : u, \partial_t u, \Delta_v u - v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}.}$$

The desired characterisation fails.

Kolmogorov equation

What is the solution space Z?

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = 0 \end{cases}$$

Parabolic ansatz

$$\cancel{Z = \{u : u, \partial_t u, \Delta_v u - v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}}$$

The desired characterisation fails.
Indeed: $\sigma(\Delta_v - v \cdot \nabla_x) = \{\lambda \in \mathbb{C} \colon \mathrm{Re}\lambda \leq 0\}$
(Metafune '01, Fornaro, Metafune, Pallara & Schnaubelt '22).

Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = 0 \end{cases}$$

The solution space is

$$Z = \{u : \; u, \; \partial_t u + v \cdot \nabla_x u, \; \Delta_v u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}.$$

Kinetic maximal L^p-regularity

Definition (*simplified*):

We say that a linear operator $A\colon D(A) \subset L^p(\mathbb{R}^{2n}) \to L^p(\mathbb{R}^{2n})$ admits kinetic maximal L^p-regularity if for all $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$ there exists a unique distributional solution

$$u \in Z = \{w \,:\; w,\, \partial_t w + v \cdot \nabla_x w,\, Aw \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

of the Cauchy problem

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = Au + f \\ u(0) = 0 \end{cases}$$

with

$$\|u\|_p + \|\partial_t u + v \cdot \nabla_x u\|_p + \|Au\|_p \leq C \|f\|_p$$

for some constant $C = C(T,p) > 0$.

Kolmogorov equation

Fundamental solution (Kolmogorov '34):

$$\Gamma(t,x,v) = \frac{c_n}{t^{2n}} \exp\left(-\frac{1}{t}\,|v|^2 + \frac{3}{t^2}\langle v, x\rangle - \frac{3}{t^3}\,|x|^2 \right).$$

Kolmogorov equation

Fundamental solution (Kolmogorov '34):

$$\Gamma(t,x,v) = \frac{c_n}{t^{2n}} \exp\left(-\frac{1}{t}\left|v\right|^2 + \frac{3}{t^2}\langle v, x\rangle - \frac{3}{t^3}\left|x\right|^2\right).$$

Solution of Kolmogorov equation with $g = 0$ is given by

$$u(t,x,v) = \int_0^t \int_{\mathbb{R}^{2n}} \Gamma(t-s, x-y-(t-s)w, v-w) f(s,y,w) \mathrm{d}(y,w) \mathrm{d}s.$$

Kolmogorov equation

Fundamental solution (Kolmogorov '34):

$$\Gamma(t,x,v) = \frac{c_n}{t^{2n}} \exp\left(-\frac{1}{t}\left|v\right|^2 + \frac{3}{t^2}\langle v, x\rangle - \frac{3}{t^3}\left|x\right|^2\right).$$

Solution of Kolmogorov equation with $g = 0$ is given by

$$u(t,x,v) = \int_0^t \int_{\mathbb{R}^{2n}} \Gamma(t-s, x-y-(t-s)w, v-w) f(s,y,w)\mathrm{d}(y,w)\mathrm{d}s.$$

Singular integral on homogeneous group (Folland & Stein '74):

$$\left\|\partial_t u + v \cdot \nabla_x u\right\|_p + \left\|\Delta_v u\right\|_p \leq C \left\|f\right\|_p.$$

For every $f \in L^p((0,T); L^p(\mathbb{R}^{2n}))$ there exists a unique solution $u \in Z$ of the Kolmogorov equation.

Kolmogorov equation

Theorem (Folland et al. '74, Bramanti et al. '10, Dong et al. '22):

For all $p \in (1, \infty)$, the operator $\Delta_v \colon H^{2,p}_v(\mathbb{R}^{2n}) \to L^p(\mathbb{R}^{2n})$ admits kinetic maximal $L^p(L^p)$-regularity for all $p \in (1, \infty)$.

Kinetic trace

Temporal trace $u(t)$ is well-defined. In particular

$$\left\{u\colon u, \partial_t u + v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\right\} \hookrightarrow C([0,T]; L^p(\mathbb{R}^{2n})).$$

Kinetic trace

Temporal trace $u(t)$ is well-defined. In particular

$$\left\{u \colon u, \partial_t u + v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\right\} \hookrightarrow C([0,T]; L^p(\mathbb{R}^{2n})).$$

The trace space of Z is defined as

$$X_\gamma = \{g \colon \exists u \in Z \text{ with } u(0) = g\}$$

$$\|g\|_{X_\gamma} = \inf_{\substack{u \in Z \\ u(0)=g}} \|u\|_Z.$$

Moreover

$$Z \hookrightarrow C([0,T]; X_\gamma).$$

Kolmogorov equation

Theorem (N. & Zacher '22):

For the Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

there exists a unique solution

$$u \in Z = \{w \, : \; w, \, \partial_t w + v \cdot \nabla_x w, \, \Delta_v w \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

if and only if

(i) $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$

(ii) $g \in X_\gamma$.

Moreover, $u \in C([0,T]; X_\gamma)$.

Kinetic trace

Recall:

$$X_\gamma = \{g \colon \exists u \in Z \text{ with } u(0) = g\} \text{ with } \|g\|_{X_\gamma} = \inf_{\substack{u \in Z \\ u(0)=g}} \|u\|_Z.$$

Kinetic trace

Recall:

$$X_\gamma = \{g \colon \exists u \in Z \text{ with } u(0) = g\} \text{ with } \|g\|_{X_\gamma} = \inf_{\substack{u \in Z \\ u(0)=g}} \|u\|_Z.$$

For the homogeneous problem $u = u(t, v)$

$$\begin{cases} \partial_t u = \Delta_v u + f \\ u(0) = g \end{cases} \qquad \text{(heat equation)}$$

we have $X_\gamma = B^{2(1-1/p)}_{pp,v}(\mathbb{R}^n)$.

Kinetic trace

Kinetic Regularisation (Bouchut '02):

Let $u \in L^p(\mathbb{R}^{1+2n})$ with $\partial_t u + v \cdot \nabla_x u \in L^p(\mathbb{R}^{1+2n})$ and $\Delta_v u \in L^p(\mathbb{R}^{1+2n})$. Then

$$D_x^{\frac{2}{3}} u \in L^p(\mathbb{R}^{1+2n}).$$

Here: $D_x^s = (-\Delta_x)^{s/2}$.

Kinetic trace

Kinetic Regularisation (Bouchut '02:)

Let $u \in L^p(\mathbb{R}^{1+2n})$ with $\partial_t u + v \cdot \nabla_x u \in L^p(\mathbb{R}^{1+2n})$ and $\Delta_v u \in L^p(\mathbb{R}^{1+2n})$. Then

$$D_x^{\frac{2}{3}} u \in L^p(\mathbb{R}^{1+2n}).$$

Recall the scaling: $\lambda \mapsto (\lambda^2 t, \lambda^3 x, \lambda v)$.

Here: $D_x^s = (-\Delta_x)^{s/2}$.

Kinetic Regularisation (Bouchut '02, Alexandre '12)

Let $u \in L^2(\mathbb{R}^{1+2n})$ with $\partial_t u + v \cdot \nabla_x u = f, \Delta_v u \in L^2(\mathbb{R}^{1+2n})$.

Fourier variables $(x, v) \to (k, \xi)$:

$$\partial_t \hat{u} - k \cdot \nabla_\xi \hat{u} = \hat{f} \qquad | \cdot \bar{\hat{u}} \text{ \& C.-S.}$$

$$\partial_t |\hat{u}|^2 - k \cdot \nabla_\xi |\hat{u}|^2 \lesssim |\hat{f}| \, |\hat{u}|$$

Kinetic Regularisation (Bouchut '02, Alexandre '12)

Let $u \in L^2(\mathbb{R}^{1+2n})$ with $\partial_t u + v \cdot \nabla_x u = f, \Delta_v u \in L^2(\mathbb{R}^{1+2n})$.

Fourier variables $(x, v) \to (k, \xi)$:

$$\partial_t \hat{u} - k \cdot \nabla_\xi \hat{u} = \hat{f} \qquad | \cdot \bar{\hat{u}} \text{ \& C.-S.}$$

$$\partial_t |\hat{u}|^2 - k \cdot \nabla_\xi |\hat{u}|^2 \lesssim |\hat{f}| \, |\hat{u}|$$

Method of characteristics:

$$|\hat{u}(t, k, \xi)|^2 \lesssim \int_{-\infty}^{t} |\hat{f}\,\hat{u}|(t - s, k, \xi + sk) \mathrm{d}s$$

Claim: $D_x^{2/3} u \in L^2(\mathbb{R}^{1+2n})$, i.e. $|k|^{2/3} \hat{u} \in L^2(\mathbb{R}^{1+2n})$.

Kinetic Regularisation (Bouchut '02, Alexandre '12)

For $k \in \mathbb{R}^n$ estimate

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |k|^{\frac{4}{3}} \, |\hat{u}(t,k,\xi)|^2 \, \mathrm{d}\xi \mathrm{d}t$$

$$= \int_{\mathbb{R}} \int_{|\xi| \geq |k|^{\frac{1}{3}}} |k|^{\frac{4}{3}} \, |\hat{u}(t,k,\xi)|^2 \, \mathrm{d}\xi \mathrm{d}t + \int_{\mathbb{R}} \int_{|\xi| \leq |k|^{\frac{1}{3}}} |k|^{\frac{4}{3}} \, |\hat{u}(t,k,\xi)|^2 \, \mathrm{d}\xi \mathrm{d}t$$

$$=: A + B.$$

Kinetic Regularisation (Bouchut '02, Alexandre '12)

Estimate

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |k|^{\frac{4}{3}} \left|\hat{u}(t,k,\xi)\right|^2 \mathrm{d}\xi \mathrm{d}t$$

$$= \int_{\mathbb{R}} \int_{|\xi| \geq |k|^{\frac{1}{3}}} |k|^{\frac{4}{3}} \left|\hat{u}(t,k,\xi)\right|^2 \mathrm{d}\xi \mathrm{d}t + \int_{\mathbb{R}} \int_{|\xi| \leq |k|^{\frac{1}{3}}} |k|^{\frac{4}{3}} \left|\hat{u}(t,k,\xi)\right|^2 \mathrm{d}\xi \mathrm{d}t$$

$$=: A + B.$$

Good part:

$$A \leq \int_{\mathbb{R}} \int_{|\xi| \geq |k|^{\frac{1}{3}}} |\xi|^4 \left|\hat{u}(t,k,\xi)\right|^2 \mathrm{d}\xi \mathrm{d}t.$$

Kinetic Regularisation (Bouchut '02, Alexandre '12)

$$B \lesssim \int_{\mathbb{R}} \int_{|\xi| \leq |k|^{\frac{1}{3}}} |k|^{\frac{4}{3}} \int_{-\infty}^{t} |\hat{f}\,\hat{u}|(t-s, k, \xi + sk) \mathrm{d}s \mathrm{d}\xi \mathrm{d}t$$

$$= \int_0^{\infty} \int_{\mathbb{R}} \int_{|\xi - sk| \leq |k|^{\frac{1}{3}}} |k|^{\frac{4}{3}} |\hat{f}\,\hat{u}|(t, k, \xi) \mathrm{d}\xi \mathrm{d}t \mathrm{d}s.$$

Kinetic Regularisation (Bouchut '02, Alexandre '12)

$$B \lesssim \int_{\mathbb{R}} \int_{|\xi| \le |k|^{\frac{1}{3}}} |k|^{\frac{4}{3}} \int_{-\infty}^{t} |\hat{f}\hat{u}|(t-s, k, \xi + sk) \mathrm{d}s \mathrm{d}\xi \mathrm{d}t$$

$$= \int_0^{\infty} \int_{\mathbb{R}} \int_{|\xi - sk| \le |k|^{\frac{1}{3}}} |k|^{\frac{4}{3}} |\hat{f}\hat{u}|(t, k, \xi) \mathrm{d}\xi \mathrm{d}t \mathrm{d}s.$$

Note $||\xi| - s\,|k|| \le |\xi - sk| \le |k|^{\frac{1}{3}}$ or $\left|\frac{|\xi|}{|k|} - s\right| \le |k|^{-\frac{2}{3}}$ hence

$$\lesssim \int_0^{\infty} \int_{\mathbb{R}} |k|^{\frac{2}{3}} |\hat{f}\hat{u}|(t, k, \xi) \mathrm{d}\xi \mathrm{d}t \mathrm{d}s$$

$$\lesssim \int_0^{\infty} \int_{\mathbb{R}} \varepsilon\, |k|^{\frac{4}{3}} |\hat{u}(t, k, \xi)|^2 + C_\varepsilon |\hat{f}(t, k, \xi)|^2 \mathrm{d}\xi \mathrm{d}t \mathrm{d}s.$$

Integrate in k and absorb the ε to finish the proof.

Kinetic trace

Recall:

$$X_\gamma = \{g\colon \exists u \in Z \text{ with } u(0) = g\} \text{ with } \|g\|_{X_\gamma} = \inf_{\substack{u\in Z\\ u(0)=g}} \|u\|_Z.$$

For the homogeneous problem $u = u(t, v)$

$$\begin{cases} \partial_t u = \Delta_v u + f \\ u(0) = g \end{cases} \qquad \text{(heat equation)}$$

we have $X_\gamma = B^{2(1-1/p)}_{pp,v}(\mathbb{R}^n)$.

Kinetic regularisation:

$$\begin{aligned} Z &= \{w \;:\; w,\, \partial_t w + v\cdot\nabla_x w,\, \Delta_v w \in L^p((0,T); L^p(\mathbb{R}^{2n}))\} \\ &= \{w \;:\; w,\, \partial_t w + v\cdot\nabla_x w,\, \Delta_v w,\, D_x^{\frac{2}{3}} w \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}. \end{aligned}$$

Kinetic trace

Theorem (N. & Zacher '22):

Let $p \in (1, \infty)$ and X_γ the trace space to

$$Z = \{u : \ u, \, \partial_t u + v \cdot \nabla_x u, \, \Delta_v u \in L^p((0, T); L^p(\mathbb{R}^{2n}))\}.$$

Then

$$X_\gamma \cong B_{pp,x}^{\frac{2}{3}(1-\frac{1}{p})}(\mathbb{R}^{2n}) \cap B_{pp,v}^{2(1-\frac{1}{p})}(\mathbb{R}^{2n}).$$

Kolmogorov equation

Theorem (N. & Zacher '22):

For the Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

there exists a unique solution

$$u \in Z = \{w : \; w, \, \partial_t w + v \cdot \nabla_x w, \, \Delta_v w \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

if and only if

(i) $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$

(ii) $g \in X_\gamma = B_{pp,x}^{\frac{2}{3}(1-\frac{1}{p})}(\mathbb{R}^{2n}) \cap B_{pp,v}^{2(1-\frac{1}{p})}(\mathbb{R}^{2n})$.

Moreover, $u \in C([0,T]; X_\gamma)$.

Fractional Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\beta/2} u + f \\ u(0) = g. \end{cases}$$

Theorem (Chen & Zhang '18; Huang, Menozzi & Priola '19):

For $\beta \in (0,2)$ the operator $-(-\Delta_v)^{\beta/2}\colon H_v^{\beta,p}(\mathbb{R}^{2n}) \to L^p(\mathbb{R}^{2n})$ admits kinetic maximal $L^p(L^p)$-regularity for all $p \in (1,\infty)$.

Theorem (N. & Zacher '22):

$$X_\gamma \cong B_{pp,x}^{\frac{\beta}{\beta+1}(1-\frac{1}{p})}(\mathbb{R}^{2n}) \cap B_{pp,v}^{\beta(1-\frac{1}{p})}(\mathbb{R}^{2n})$$

Temporal weights

Replace $L^p((0,T);X)$ with

$$L^p_\mu((0,T);X) = \{u\colon t^{1-\mu}u \in L^p((0,T);X)\}$$

with $\mu \in (1/p, 1]$ (Muckenhoupt weight, Prüss & Simonett '04).

Temporal weights

Replace $L^p((0,T);X)$ with

$$L^p_\mu((0,T);X) = \{u\colon t^{1-\mu}u \in L^p((0,T);X)\}$$

with $\mu \in (1/p, 1]$ (Muckenhoupt weight, Prüss & Simonett '04).

Advantages:

- Theorem (N. & Zacher '22):
  Kinetic maximal L^p_μ-regularity is independent of $\mu \in (1/p, 1]$.
- For (fractional) Kolmogorov equation:

$$X_{\gamma,\mu} \cong B^{\frac{\beta}{\beta+1}(\mu-\frac{1}{p})}_{pp,x}(\mathbb{R}^{2n}) \cap B^{\beta(\mu-\frac{1}{p})}_{pp,v}(\mathbb{R}^{2n}).$$

- They allow to observe instantaneous regularisation.

Different base spaces

Theorem(s) (N. & Zacher '22,'23):

- Kinetic maximal $L^p(L^q)$-regularity for $-(-\Delta_v)^{\beta/2}$ with $p, q \in (1, \infty)$.

Different base spaces

Theorem(s) (N. & Zacher '22,'23):

- Kinetic maximal $L^p(L^q)$-regularity for $-(-\Delta_v)^{\beta/2}$ with $p, q \in (1, \infty)$.

Consider $[\Gamma u](t, x, v) = u(t, x + tv, v)$. Then if

$$\partial_t u + v \cdot \nabla_x u = \Delta_v u + f$$

we have that $w = \Gamma u$ solves

$$\partial_t w = \Gamma \Delta_v \Gamma^{-1} w + \Gamma f = (\nabla_v - t\nabla_x)^2 w + \Gamma f.$$

Non-autonomous degenerate PDE (Hieber & Monniaux '00).

Different base spaces

Theorem(s) (N. & Zacher '22,'23):

- Kinetic maximal $L^p(L^q)$-regularity for $-(-\Delta_v)^{\beta/2}$ with $p, q \in (1, \infty)$.

- Kinetic maximal $L^p(L^q_{j,k})$-regularity for Δ_v with $p, q \in (1, \infty)$ and $j, k \in \mathbb{R}$ where $L^q_{j,k}$ is weighted with $(1 + |v|)^j$ and $(1 + |x| + |v|)^k$.

Different base spaces

Theorem(s) (N. & Zacher '22,'23):

- Kinetic maximal $L^p(L^q)$-regularity for $-(-\Delta_v)^{\beta/2}$ with $p, q \in (1, \infty)$.

- Kinetic maximal $L^p(L^q_{j,k})$-regularity for Δ_v with $p, q \in (1, \infty)$ and $j, k \in \mathbb{R}$ where $L^q_{j,k}$ is weighted with $(1 + |v|)^j$ and $(1 + |x| + |v|)^k$.

- Kinetic maximal $L^p(X^{s,q}_\beta)$-regularity for $-(-\Delta_v)^{\beta/2}$

$$X^{s,q}_\beta = \left\{ f \in \mathcal{S}' : \ \left(1 + |\xi|^\beta + |k|^{\frac{\beta}{\beta+1}}\right)^s \mathcal{F}(f) \in L^q \right\}$$

  with $p, q \in (1, \infty)$, $s \geq 0$ and $p \in (1, \infty)$, $q = 2$, $s \geq -1/2$.

Including a characterisation of the trace space.

Kolmogorov equation with variable coefficients

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = a(t,x,v) \colon \nabla_v^2 u + f \\ u(0) = g \end{cases}$$

Under which assumptions on the coefficient $a(t,x,v)$ do we obtain kinetic maximal L^p-regularity?

Kolmogorov equation with variable coefficients

Theorem (Bramanti et al. '13, N. & Zacher '22):

Let $a = a(t, x, v) \in L^\infty([0, T] \times \mathbb{R}^{2n}; \mathrm{Sym}(n))$ with $\lambda |\xi|^2 \leq \langle a(t, x, v)\xi, \xi\rangle$ for all (t, x, v) and $\xi \in \mathbb{R}^n$.

Kolmogorov equation with variable coefficients

Theorem (Bramanti et al. '13, N. & Zacher '22):

Let $a = a(t,x,v) \in L^\infty([0,T] \times \mathbb{R}^{2n}; \mathrm{Sym}(n))$ with $\lambda |\xi|^2 \leq \langle a(t,x,v)\xi, \xi\rangle$ for all (t,x,v) and $\xi \in \mathbb{R}^n$. Suppose

$\forall \varepsilon > 0\colon \exists \delta > 0$ such that $|t-s| + |x - y - (t-s)v| + |v - w| < \delta$ implies $|a(t,x,v) - a(s,y,w)| < \varepsilon \quad (\mathrm{BUC_{kin}})$

OR

$\forall \varepsilon > 0\colon \exists \delta > 0$ such that $|t-s| + |x - y| + |v - w| < \delta$ implies $|a(t,x,v) - a(s,y,w)| < \varepsilon \quad (\mathrm{BUC})$.

Kolmogorov equation with variable coefficients

Theorem (Bramanti et al. '13, N. & Zacher '22):

Let $a = a(t,x,v) \in L^\infty([0,T] \times \mathbb{R}^{2n}; \mathrm{Sym}(n))$ with $\lambda |\xi|^2 \leq \langle a(t,x,v)\xi, \xi \rangle$ for all (t,x,v) and $\xi \in \mathbb{R}^n$. Suppose

$\forall \varepsilon > 0\colon \exists \delta > 0$ such that $|t-s| + |x-y-(t-s)v| + |v-w| < \delta$ implies $|a(t,x,v) - a(s,y,w)| < \varepsilon \quad (\mathrm{BUC_{kin}})$

OR

$\forall \varepsilon > 0\colon \exists \delta > 0$ such that $|t-s| + |x-y| + |v-w| < \delta$ implies $|a(t,x,v) - a(s,y,w)| < \varepsilon \quad (\mathrm{BUC})$.

Then the family of operators

$$A(t) = a(t,x,v) : \nabla_v^2 : H^{2,p}_{v,j,k}(\mathbb{R}^{2n}) \to L^p_{j,k}(\mathbb{R}^{2n})$$

admits kinetic maximal $L^p_\mu(L^q_{j,k})$-regularity.

Fractional Kolmogorov equation with variable density

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\beta/2} u + f \\ u(0) = g \end{cases}$$

Recall

$$-(-\Delta_v)^{\beta/2} u = c_{n,\beta} \ \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t,x,v+h) - u(t,x,v)}{|h|^{n+\beta}} \mathrm{d}h$$

for $\beta \in (0,2)$ and $c_{n,\beta>0}$.

Fractional Kolmogorov equation with variable density

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\beta/2} u + f \\ u(0) = g \end{cases}$$

Consider

$$-(-\Delta_v)^{\beta/2} u = c_{n,\beta} \ \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t,x,v+h) - u(t,x,v)}{|h|^{n+\beta}} a(t,x,v,h) \mathrm{d}h$$

for $\beta \in (0,2)$ and $c_{n,\beta>0}$.

Fractional Kolmogorov equation with variable density

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = A(t)u + f \\ u(0) = g \end{cases}$$

Consider

$$A(t)u = \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(t,x,v+h) - u(t,x,v)}{|h|^{n+\beta}} a(t,x,v,h) \mathrm{d}h$$

for $\beta \in (0,2)$.

Fractional Kolmogorov equation with variable density

Theorem (N. '22):

Let $\alpha \in (0,1)$ and $a = a(t,x,v,h) \in L^\infty([0,T] \times \mathbb{R}^{3n})$ symmetric in h with $0 < \lambda \leq a \leq \Lambda$ and

$$\sup \frac{|a(t,x,v,h) - a(s,y,w,h)|}{|t-s|^\alpha + |x-y-(t-s)v|^\alpha + |v-w|^\alpha} < \infty.$$

Then, the family of operators

$$A(t)u = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t,x,v+h) - u(t,x,v)}{|h|^{n+\beta}} a(t,x,v,h) \mathrm{d}h$$

admits kinetic maximal $L^p_\mu(L^p)$-regularity for all $p > \frac{n}{\alpha}$, $\mu \in (1/p, 1]$.

Same trace space as for $-(-\Delta_v)^{\beta/2}$.

Application to quasilinear equations

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = A(u)u + F(u) \\ u(0) = g \end{cases}$$

Application to quasilinear equations

Think of X as $L^q_{j,k}(\mathbb{R}^{2n})$ and let $D \subset X$. Seek solutions in

$$Z = \{u \colon u, \partial_t u + v \cdot \nabla_x u \in L^p_\mu((0,T);X)\} \cap L^p_\mu((0,T);D)$$

of

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = A(u)u + F(u) \\ u(0) = g \end{cases}$$

where

- $g \in X_{\gamma,\mu}$
- $A \colon X_{\gamma,\mu} \to \mathcal{B}(D,X)$
- $F \colon X_{\gamma,\mu} \to X$.

Application to quasilinear equations

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = A(u)u + F(u) \\ u(0) = g \end{cases} \tag{1}$$

Theorem (N. & Zacher '22):

Assume that

– $(A, F) \in C^{1-}_{\mathrm{loc}}(X_{\gamma,\mu}; \mathcal{B}(D, X) \times X)$

– $A(g)$ admits kinetic maximal $L^p_\mu(X)$-regularity.

Then there exists $T = T(g)$ and $\varepsilon = \varepsilon(g) > 0$ such that (1) admits a unique solution in Z for all $h \in \overline{B_\varepsilon(g)}^{X_{\gamma,\mu}}$.
Moreover, solutions depend continuously on the initial datum.

Here: $X = X^{s,q}_{\beta,j,k}$, $D \subset X$ and $Z = \mathcal{T}^p_\mu((0,T); X) \cap L^p_\mu((0,T); D)$.

A kinetic toy model

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = M(u)\Delta_v u \\ u(0) = g \end{cases} \tag{1}$$

with the local density $M(u)(t,x) = \int_{\mathbb{R}^n} u(t,x,v)\mathrm{d}v$.
(Villani '00, Liao et al. '18, Mouhot & Imbert '21, Anceschi & Zhu '21)

A kinetic toy model

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = M(u)\Delta_v u \\ u(0) = g \end{cases} \tag{1}$$

with the local density $M(u)(t,x) = \int_{\mathbb{R}^n} u(t,x,v)\mathrm{d}v$.

Theorem (N. & Zacher '23):

Let $j > n$, $\lambda > 0$, $p, q \in (1, \infty)$, $\mu \in (1/p, 1]$ with $\mu - 1/p > 2n/q$. Then for every $g \in {}^{\mathrm{kin}}B^{\mu-1/p,2}_{qp,j}(\mathbb{R}^{2n})$ with $M(g) \geq \lambda$ there exists a time $T = T(g)$ such that (1) admits a unique solution

$$u \in \mathcal{T}^p_\mu((0,T); L^q_j(\mathbb{R}^{2n})) \cap L^p_\mu((0,T); H^{2,q}_{v,j}(\mathbb{R}^{2n})).$$

Note that: ${}^{\mathrm{kin}}B^{\mu-1/p,2}_{qp,j}(\mathbb{R}^{2n}) \hookrightarrow C_{0,j}(\mathbb{R}^{2n})$.

A kinetic toy model

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = M(u)\Delta_v u \\ u(0) = g \end{cases} \tag{1}$$

Theorem (N. & Zacher '23):

Assumptions as before. Let u be the solution to (1) with initial value $0 \leq g \in {}^{\mathrm{kin}}B^{\mu-1/p,2}_{qp,j}(\mathbb{R}^{2n})$ extended to $[0, T_{\max})$. If there exist $0 < M_0 < M_1$ such that

$$M_0 \leq M(u)(t,x) \leq M_1 \text{ for all } (t,x) \in [0, T_{\max}) \times \mathbb{R}^n$$

then $T_{\max} = \infty$.

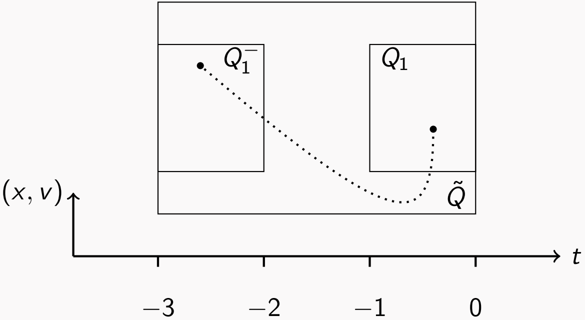
Conditional global existence

Geometry?

Trajectories and De Giorgi-Nash-Moser theory

L. N and R. Zacher. *A trajectorial interpretation of Moser's proof of the Harnack inequality*. To appear in Annali della Scuola Normale Superiore di Pisa - Classe di Scienze (2023).

L. N and R. Zacher. *On a kinetic Poincaré inequality and beyond*. Preprint. arXiv:2212.03199 (2022).



Bibliography

- L. N., R. Zacher. *Kinetic maximal L^2-regularity for the (fractional) Kolmogorv equation*. Journal of Evolution Equations 21 (2021).
- L. N., R. Zacher. *Kinetic maximal L^p-regularity with temporal weights and application to quasilinear kinetic diffusion equations*. Journal of Differential Equations 307 (2022).
- L. N. *Kinetic maximal $L^p_\mu(L^p)$-regularity for the fractional Kolmogorov equation with variable density*. Nonlinear Analysis (2022).
- L. N. *Analytic aspects of kinetic partial differential equations*. PhD thesis, Ulm University (2023).

lukasniebel.github.io