Bubbles and drops in inviscid fluids

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1. Experiments

1. Experiments (videos, links on the last slide)
2. Two-phase Euler equations with surface tension
3. 3D Traveling wave solutions (existence of close-to-spherical solutions)
4. 2D Stationary solutions (rigidity of the circular solution)

2. Two-phase Euler equations with surface tension

Euler equations

Velocity field of the fluid $U = U(t,x)\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ solution to

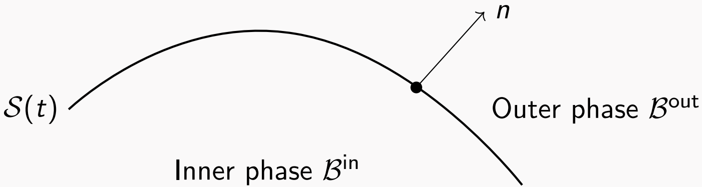
$$\begin{aligned} \partial_t U + (U \cdot \nabla) U + \nabla P &= 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}^3 \\ \nabla \cdot U &= 0 \qquad \text{in } \mathbb{R}^3 \end{aligned}$$

where $P\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is the pressure.

Two-phase Euler equations

Velocity field of the fluid $U\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ solution to

$$\begin{aligned} \rho(\partial_t U + (U \cdot \nabla) U) + \nabla P &= 0 && \text{in } \mathbb{R} \times \mathbb{R}^3 \\ \nabla \cdot U &= 0 && \text{in } \mathbb{R} \times \mathbb{R}^3 \\ [\![U \cdot n]\!] &= 0 && \text{on } \mathcal{S}(t) \end{aligned}$$



Two-phase Euler equations

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where

- $P\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ is the pressure
- $\mathcal{S}(t)$ is the interface separating the inner $\mathcal{B}^{\mathrm{in}}(t)$ and outer $\mathcal{B}^{\mathrm{out}}(t)$ fluid domain
- $\rho(t) = \rho^{\mathrm{in}}\mathbb{1}_{\mathcal{B}^{\mathrm{in}}(t)} + \rho^{\mathrm{out}}\mathbb{1}_{\mathcal{B}^{\mathrm{out}}(t)}$ for $\rho^{\mathrm{in}}, \rho^{\mathrm{out}} \geq 0$, is the density function
- $[\![f]\!] = f^{\mathrm{out}} - f^{\mathrm{in}}$, the jump of a quantity f across the interface.

Two-phase Euler equations with surface tension

Velocity field of the fluid $U \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ solution to

$$
\begin{aligned}
\rho(\partial_t U + (U \cdot \nabla) U) + \nabla P &= 0 && \text{in } \mathbb{R} \times \mathbb{R}^3 \\
\nabla \cdot U &= 0 && \text{in } \mathbb{R} \times \mathbb{R}^3 \\
[\![P]\!] &= \sigma H && \text{on } \mathcal{S}(t) \\
[\![U \cdot n]\!] &= 0 && \text{on } \mathcal{S}(t)
\end{aligned}
$$

where

- we take into consideration the Young-Laplace law
- H is the mean curvature ($H = 2$ for the unit ball)
- $\sigma > 0$ is the surface tension

3. 3D Traveling wave solutions

Traveling wave solutions

We make the ansatz

$$\begin{aligned} u(x) &= U(t, x_1, x_2, x_3 + Vt) - Ve_3 \\ p(x) &= P(t, x_1, x_2, x_3 + Vt) \\ \mathcal{S}(t) &= \mathcal{S} + tVe_3, \end{aligned}$$

for some speed $V \geq 0$.

Moving frame

The time-independent $u, p, \mathcal{S}$ solve the steady two-phase Euler equations

$$\begin{aligned} \rho(u\cdot\nabla)u + \nabla p &= 0 && \text{in } \mathbb{R}^3\backslash\mathcal{S}, \\ \nabla\cdot u &= 0 && \text{in } \mathbb{R}^3, \\ [\![p]\!] &= \sigma H && \text{on } \mathcal{S}, \\ u\cdot n &= 0 && \text{on } \mathcal{S}. \end{aligned}$$

with $\lim_{|x|\to\infty} u(x) = -Ve_3$.

Bernoulli laws

Bernoulli laws (for steady flows) for the inner/outer phase are

$$\frac{\rho^{\mathrm{in}}}{2}\left|u^{\mathrm{in}}\right|^2 + p^{\mathrm{in}} = \mathrm{const},$$
$$\frac{\rho^{\mathrm{out}}}{2}\left|u^{\mathrm{out}}\right|^2 + p^{\mathrm{out}} = \mathrm{const}.$$

Rewrite the interfacial condition

$$[\![P]\!] = \sigma H \ \text{ on } \mathcal{S}$$

as

$$\frac{1}{2}[\![\rho|u|^2]\!] + \sigma H = \mathrm{const} \ \text{ on } \mathcal{S}.$$

Modelling assumptions

We are interested in axisymmetric and swirl-free vector fields ($u = u(r, z)$ and azimutal component $u_\varphi = 0$).

The fluid in the outer domain is assumed to be irrotational $\operatorname{curl} u^{\mathsf{out}} = 0$.

The volume is $|\mathcal{S}| = \dfrac{4}{3}\pi R^3$.

Inner motion

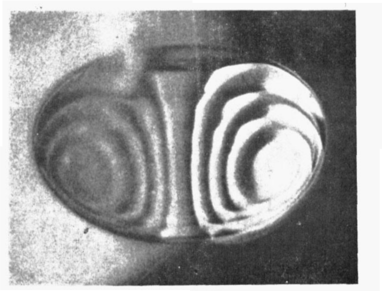


Fig 2. 60% glycerine-water drop (4% sucrose) falling in heavy white oil.
Linear magnification × 12.

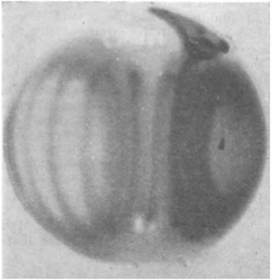


Fig. 2. Flow pattern inside water drop of diameter 8 mm moving through mineral oil at a velocity of 2·5 cm/sec

We assume vorticity distribution in the inner phase to be

$$\operatorname{curl} u^{\mathrm{in}} = \omega_a = \frac{15}{2} a \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$$

for $a \in \mathbb{R}$.

[left] K. E. Spells, Proc. Phys. Soc. B 65 541 (1952).

[right] Magarvey, R., Kalejs, J. Internal Circulations within Liquid Drops. Nature 198, 377–378 (1963).

Vector stream function

We work with the vector stream function $\psi\colon \mathbb{R}^3 \to \mathbb{R}^3$ with

$$u = \operatorname{curl} \psi - Ve_3.$$

The tangential flow and the axisymmetry no-swirl condition yields

$$\psi = \frac{V}{2} r e_\varphi \text{ on } \mathcal{S}.$$

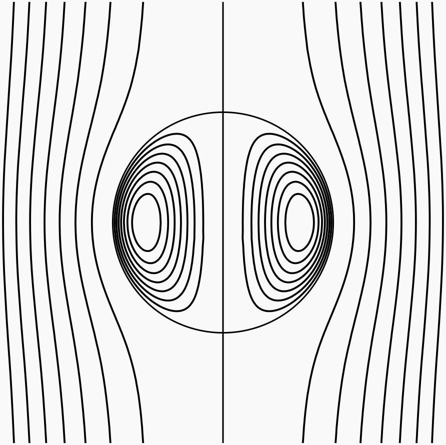
The identity $\operatorname{curl}\operatorname{curl} = \nabla\nabla \cdot - \Delta$ implies

$$-\Delta\psi = \omega_a \mathbb{1}_{\mathcal{B}^{\text{in}}} \text{ in } \mathbb{R}^3 \setminus \mathcal{S}.$$

The jump condition becomes

$$\frac{1}{2} \left[\!\left[\rho \left| \operatorname{curl} \psi - Ve_3 \right|^2 \right]\!\right] + \sigma H = \text{const on } \mathcal{S}.$$

Spherical solution with Hill's vortex core



Streamlines of ψ_S in axisymmetric coordinates

Spherical solution with Hill's vortex core

Consider $\mathcal{S}$ to be the sphere of radius R and

$$\psi_S(x) = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \cdot \begin{cases} \dfrac{3a}{4}\left(R^2 - |x|^2\right) + \dfrac{V_S}{2} & \text{for } |x| \leq R \\ \dfrac{V_S}{2}\dfrac{R^3}{|x|^3} & \text{for } |x| > R, \end{cases}$$

where V_S is the speed.

Spherical solution with Hill's vortex core

A first solution is given by $\mathcal{S}$ the sphere of radius R

$$\psi_S(x) = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \cdot \begin{cases} \frac{3a}{4}\left(R^2 - |x|^2\right) + \frac{V_S}{2} & \text{for } |x| \leq R \\ \frac{V_S}{2}\frac{R^3}{|x|^3} & \text{for } |x| > R, \end{cases}$$

where $V_S = |a|\, R^2 \sqrt{\frac{\rho^{\text{in}}}{\rho^{\text{out}}}}$ is determined such that

$$\frac{1}{2}[\![\rho|\mathrm{curl}\,\psi_S - V_S e_3|^2]\!] = \frac{9}{8R^2}\left(a^2 R^4 \rho^{\text{in}} - \rho^{\text{out}} V_S^2\right)\left(x_1^2 + x_2^2\right)$$

is constant on the sphere of radius R and thus

$$\frac{1}{2}[\![\rho|\mathrm{curl}\,\psi_S - V_S e_3|^2]\!] + \sigma H = 2\sigma R = \mathrm{const.}$$

Spherical solution with Hill's vortex core

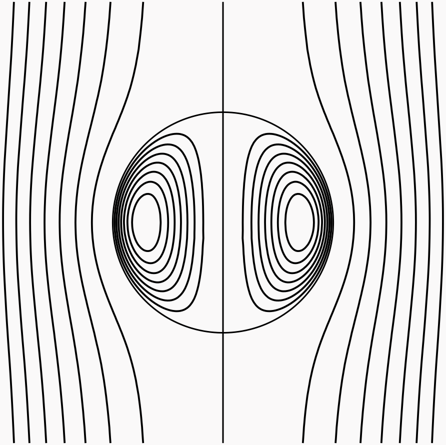
A first solution is given by $\mathcal{S}$ the sphere of radius R

$$\psi_S(x) = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \cdot \begin{cases} \frac{3a}{4}\left(R^2 - |x|^2\right) + \frac{V_S}{2} & \text{for } |x| \le R \\ \frac{V_S}{2}\frac{R^3}{|x|^3} & \text{for } |x| > R, \end{cases}$$

with $V_S = |a|\, R^2 \sqrt{\frac{\rho^{\text{in}}}{\rho^{\text{out}}}}$.

Vortex sheet, i.e. nonzero jump of $U_S \cdot \tau$ at $\mathcal{S}$, whenever $V_S \neq aR^2$.

Spherical solution with Hill's vortex core



Streamlines of ψ_S in axisymmetric coordinates

The overdetermined free boundary value problem

Given parameters $\rho^{\mathsf{in}}, \rho^{\mathsf{out}}, a, R, V$ find surface $\mathcal{S}$ and stream function ψ solution to

$$\begin{cases} -\Delta\psi = \dfrac{15}{2} a\, s \sin\theta e_\varphi \mathbb{1}_{\mathcal{B}^{\mathsf{in}}} & \text{in } \mathbb{R}^3 \setminus \mathcal{S} \\ \psi = \dfrac{V}{2} s \sin\theta e_\varphi & \text{on } \mathcal{S} \\ \dfrac{1}{2} \left[\!\left[\rho \, |\mathrm{curl}\, \psi - V e_3|^2 \right]\!\right] + \sigma H = \mathrm{const} & \text{on } \mathcal{S} \end{cases}$$

Spherical coordinates $(s, \theta, \varphi) \in [0, \infty) \times [0, \pi) \times [0, 2\pi)$.

Rescaling

Given parameters $\rho^{\text{in}}, \rho^{\text{out}}, \sigma, a, R, V$ find a surface $\mathcal{S}$ and a stream function ψ solution to

$$\begin{cases} -\Delta\psi = \dfrac{15}{2} a\, s \sin\theta e_\varphi \mathbb{1}_{\mathcal{B}^{\text{in}}} & \text{in } \mathbb{R}^3 \setminus \mathcal{S} \\ \psi = \dfrac{V}{2} s \sin\theta e_\varphi & \text{on } \mathcal{S} \\ \dfrac{1}{2}\left[\!\!\left[\rho \left|\mathrm{curl}\,\psi - V e_3\right|^2\right]\!\!\right] + \sigma H = \mathrm{const} & \text{on } \mathcal{S} \end{cases}$$

Weber number: $\mathrm{We} = \dfrac{\rho^{\text{out}} V^2 R}{\sigma}$

Vortex Weber number: $\gamma = \dfrac{\rho^{\text{in}} a^2 R^5}{\sigma}$

Splitting

Rescale to $R = 1$ and decompose

$$\psi = \left(a\psi^{\mathrm{in}} + \frac{V}{2} s \sin\theta \, e_\varphi \right) \mathbb{1}_{\mathcal{B}^{\mathrm{in}}} + V\psi^{\mathrm{out}} \mathbb{1}_{\mathcal{B}^{\mathrm{out}}},$$

with $\psi^{\mathrm{in}} \colon \mathcal{B}^{\mathrm{in}} \to \mathbb{R}^3$ solution to

$$\begin{cases} -\Delta\psi^{\mathrm{in}} = \frac{15}{2} s \sin\theta \, e_\varphi & \text{in } \mathcal{B}^{\mathrm{in}}, \\ \psi^{\mathrm{in}} = 0 & \text{on } \mathcal{S}, \end{cases}$$

and $\psi^{\mathrm{out}} \colon \mathcal{B}^{\mathrm{out}} \to \mathbb{R}^3$ vanishing at infinity and solving

$$\begin{cases} -\Delta\psi^{\mathrm{out}} = 0 & \text{in } \mathcal{B}^{\mathrm{out}}, \\ \psi^{\mathrm{out}} = \frac{1}{2} s \sin\theta \, e_\varphi & \text{on } \mathcal{S}. \end{cases}$$

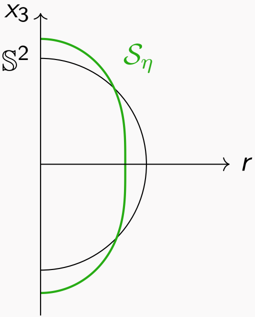
Jump condition: $\dfrac{\gamma}{2} |\operatorname{curl} \psi^{\mathrm{in}}|^2 - \dfrac{\mathrm{We}}{2} |\operatorname{curl} \psi^{\mathrm{out}} - e_3|^2 + H = \mathrm{const}$ on $\mathcal{S}$.

Perturbation of the spherical solution

For a shape function $\eta \in \mathsf{H}^{\beta}(\mathbb{S}^2)$ we consider

$$\mathcal{S}_\eta = \left\{(1+\eta(x))x : x \in \mathbb{S}^2\right\}.$$

In axisymmetric coordinates:



Perturbation of the spherical solution

For a shape function $\eta \in \mathsf{H}^{\beta}(\mathbb{S}^2)$ we consider

$$\mathcal{S}_\eta = \big\{(1+\eta(x))x : x \in \mathbb{S}^2\big\},$$

with $\mathcal{B}_\eta^{\mathsf{in}}$ and $\mathcal{B}_\eta^{\mathsf{out}}$ well-defined if $\eta > -1$.

We impose

- axi-symmetry $\eta = \eta(\theta)$, and
- reflection invariance across the reference plane, $\eta(\frac{\pi}{2}-\theta) = \eta(\frac{\pi}{2}+\theta)$

and write $\mathsf{H}^{\beta}_{\mathrm{sym}}(\mathbb{S}^2)$ for that subspace.

Set $\mathcal{M}^\beta = \{\eta \in \mathsf{H}^{\beta}_{\mathrm{sym}}(\mathbb{S}^2) : \left|\mathcal{B}_\eta^{\mathsf{in}}\right| = \frac{4}{3}\pi \text{ and } \|\eta\|_{\mathsf{H}^\beta} \le c_0\}$ for $c_0 > 0$ small.

Perturbative ansatz

We introduce the functional $\mathcal{F}\colon \mathbb{R} \times \mathbb{R} \times \mathcal{M}^{\alpha+2} \to \mathsf{H}^{\alpha}_{\mathrm{sym}}(\mathbb{S}^2)/_{\mathrm{const}}$ as

$$\mathcal{F}(\gamma, \mathrm{We}, \eta) = \frac{\gamma}{2} \left|(\mathrm{curl}\, \psi_\eta^{\mathsf{in}}) \circ \chi_\eta\right|^2 - \frac{\mathrm{We}}{2} \left|(\mathrm{curl}\, \psi_\eta^{\mathsf{out}}) \circ \chi_\eta - e_3\right|^2 + H_\eta \circ \chi_\eta$$

where $\chi_\eta = (1 + \eta(x))x$.

Goal: find We, γ and η such that

$$\mathcal{F}(\gamma, \mathrm{We}, \eta) = \mathrm{const}.$$

Spherical solution:

$$\mathcal{F}(\gamma, \gamma, 0) = 2 = \mathrm{const}.$$

$(1)\ \mathcal{F}(\gamma, \mathrm{We}, \eta) = \mathrm{const}$

Theorem (MNS '25):

Let $\beta > 2$. There exists $c_0 = c_0(\beta) > 0$ and an increasing sequence $\Gamma = (\gamma_k)_{k\in\mathbb{N}}$ of positive numbers diverging to infinity with:

(A) For any $\gamma \in [0, \infty) \setminus \Gamma$ and any We close to but different from γ, there exists a unique nontrivial (smooth) solution $\eta = \eta(\gamma, \mathrm{We}) \in \mathcal{M}^\beta$ to the jump equation (1).

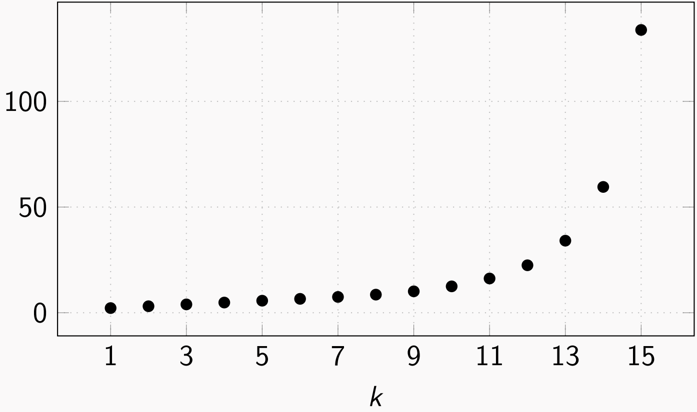
If $\gamma = \varepsilon\delta^{\mathsf{in}}$ and $\mathrm{We} = \varepsilon\delta^{\mathsf{out}}$ for two nonnegative constants $\delta^{\mathsf{in}} \neq \delta^{\mathsf{out}}$ and a small parameter ε, we have the asymptotic expansion

$$\eta_\varepsilon = \varepsilon\frac{3}{32}(\delta^{\mathsf{in}} - \delta^{\mathsf{out}})\left(3\cos^2\theta - 1\right) + o(\varepsilon) \text{ as } \varepsilon \to 0.$$

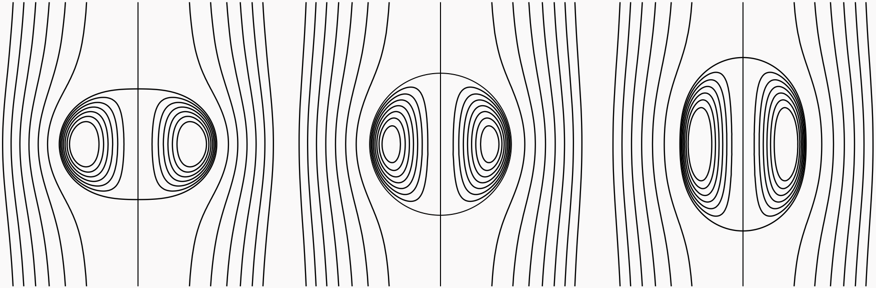
(B) For any $k \in \mathbb{N}$, there exists a unique local curve $s \mapsto \gamma(s)$ passing through γ_k and there are associated nontrivial (smooth) shape functions $\eta(s) \in \mathcal{M}^\beta$ such that the equation (1) is solved at $(\gamma(s), \gamma(s), \eta(s))$.

Bifurcation values

k	1	2	3	4	5	6
γ_k	2.20516	3.07529	3.94492	4.81679	5.69137	6.56836

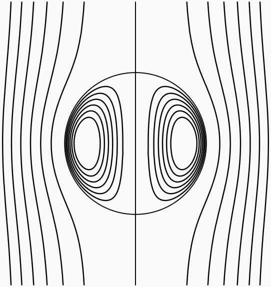


Theorem (A)

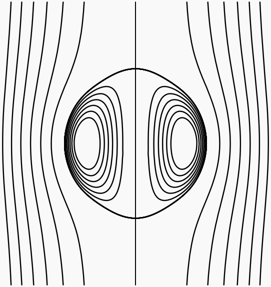


$\mathrm{We} > \gamma$ $\mathrm{We} = \gamma$ $\mathrm{We} < \gamma$

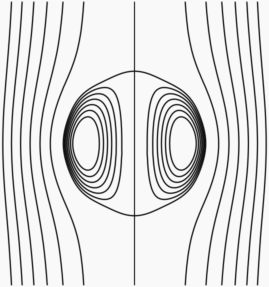
Theorem (B)



$\mathrm{We} = \gamma$

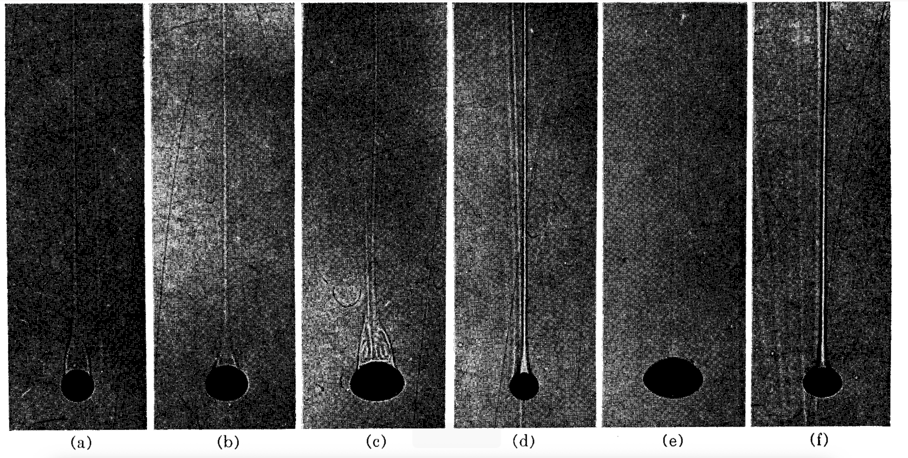


γ_1



γ_2

Air bubbles



Experimental observations

Droplet Motion in Purified Systems, S. Winnikow and B. T. Chao (1966)

Corollary (MNS ’25):

There exist values of γ close to the bifurcation set Γ for which non-spherical steady vortex solutions with $\mathrm{We} = \gamma$ exist. In particular, for these values, the spherical vortex is non-unique.

In the one fluid setting ($\rho^{\mathsf{in}} = \rho^{\mathsf{out}}$) without surface tension ($\sigma = 0$) the spherical solution with Hill’s vortex core is unique up to translations (Amick-Fraenkel ’86).

Physics literature

- Moore '58 derives the formal asymptotics of the shape for small Weber numbers neglecting the internal motion ($\rho^{\text{in}} = 0$).
- Harper '72 explains that the inner circulation can be approximated by Hill's vortex core.
- Pozrikidis '89 provides numerical evidence of the bifurcation branch and finds approximations for γ_1, γ_2.

Mathematics literature

- Crowdy-Wegmann '00 investigate two-dimensional vortex sheets
- Meyer-Seis '24 construct bubble rings
- Baldi-La Manna-La Scala '25 construct rotating solutions of close-to-spherical shape
- Murgante-Roulley-Scrobogna '25 investigate the dynamics of of close-to-spherical vortex sheets

$(1)\ \mathcal{F}(\gamma, \mathrm{We}, \eta) = \mathrm{const}$

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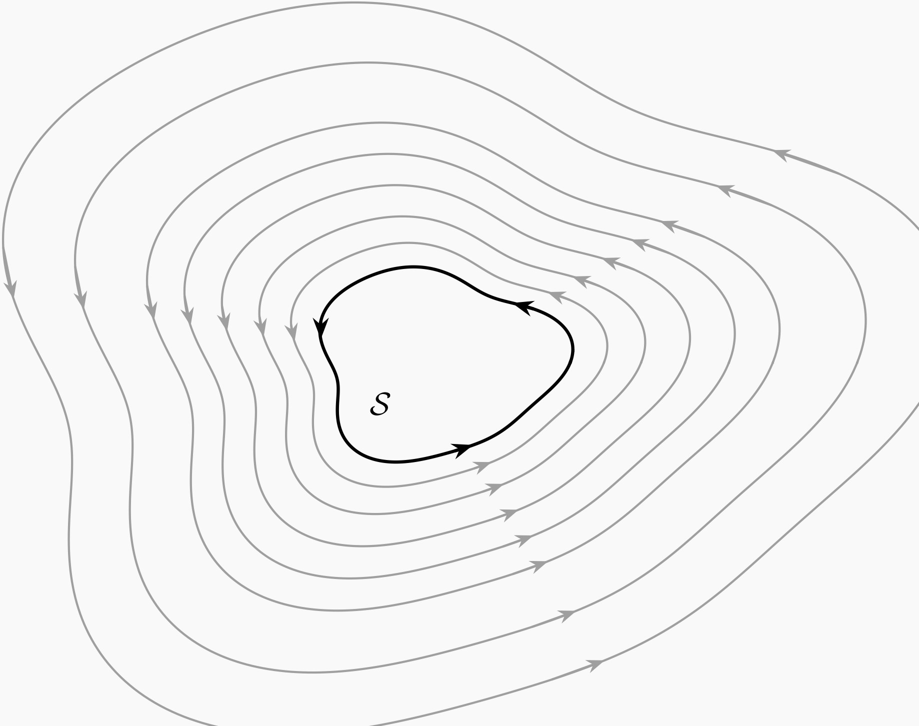
Idea of the proof

Calculate:

$$\langle \mathrm{D}_\eta \mathcal{F}(\gamma, \gamma, \eta)|_{\eta=0}, \delta\eta \rangle = \frac{9}{2}\gamma \sin\theta \, e_\varphi \cdot (2\mathrm{Id} - \Lambda)(\sin\theta \, \delta\eta \, e_\varphi) - (\Delta_{\mathbb{S}^2} + 2\mathrm{Id}) \, \delta\eta,$$

where Λ is the Dirichlet-to-Neumann map for the Laplacian on the unit ball in $\mathbb{R}^3$.

4. 2D Stationary solutions


$\mathcal{S}$

Overdetermined free boundary value problem

Find $\psi\colon \mathcal{B}^{\mathsf{out}} \subset \mathbb{R}^2 \to \mathbb{R}$ such that

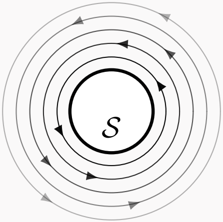
$$\begin{aligned} -\Delta\psi &= 0 && \text{in } \mathcal{B}^{\mathsf{out}} \\ \psi &= C_0 && \text{on } \mathcal{S} \\ \psi(x) &= \log|x| + O(1) && \text{as } |x| \to \infty \\ -\frac{1}{2}\mathrm{We}\,(\partial_n\psi)^2 + H &= \lambda && \text{on } \mathcal{S} \end{aligned} \tag{O}$$

with

- Weber number $\mathrm{We} \geq 0$
- constants $C_0, \lambda \in \mathbb{R}$
- closed Jordan curve $\mathcal{S} \in \mathrm{C}^{1,1}$
- $\mathcal{S}$ partitions $\mathbb{R}^2$ into $\mathcal{B}^{\mathsf{in}}$ and $\mathcal{B}^{\mathsf{out}}$ with $\left|\mathcal{B}^{\mathsf{in}}\right| = \pi$

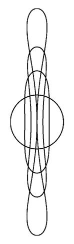
Circular solution

- $\mathcal{S} = \partial B_1(0)$
- $H = 1$
- $\psi = C_0 + \log|x|$, hence $\partial_n \psi = 1$ on $\mathcal{S}$
- $\lambda = -\frac{1}{2}\mathrm{We} + 1$

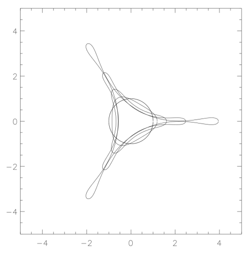


Non-circular solutions

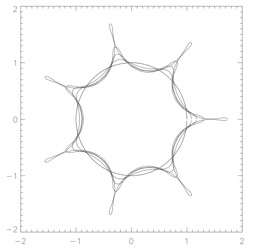
exist for all $\mathrm{We} \in \{3, 4, 5, 6, \dots\}$.



$\mathrm{We} = 3$ [1]



$\mathrm{We} = 4$ [2]



$\mathrm{We} = 8$ [2]

[1] D. Crowdy, *Circulation-induced shape deformations of drops and bubbles: Exact two-dimensional models*, Phys. Fluids **11** 2836 (1999), 2836–2845.

[2] R. Wegmann and D. Crowdy, *Shapes of two-dimensional bubbles deformed by circulation*, Nonlinearity **13** (2000), 2131–2141.

Global rigidity of the circular solution

Theorem (N., 2025)

Let $(\mathcal{S}, \psi, C_0, \mathrm{We}, \lambda)$ be a solution to (O), which is not a circle, then $\mathrm{We} > 2$.

Proof

Integrate the jump condition $-\frac{1}{2}\mathrm{We}\,(\partial_n\psi)^2 + H = \lambda$ on $\mathcal{S}$

multiplied by 1

$$-\frac{\mathrm{We}}{2}\int_{\mathcal{S}}(\partial_n\psi)^2\,\mathrm{d}\mathcal{H}^1 + \int_{\mathcal{S}} H\,\mathrm{d}\mathcal{H}^1 = \lambda\mathcal{H}^1(\mathcal{S})$$

multiplied by $(x\cdot n)$

$$-\frac{\mathrm{We}}{2}\int_{\mathcal{S}}(x\cdot n)(\partial_n\psi)^2\,\mathrm{d}\mathcal{H}^1 + \int_{\mathcal{S}} H(x\cdot n)\,\mathrm{d}\mathcal{H}^1 = \lambda\int_{\mathcal{S}} x\cdot n\,\mathrm{d}\mathcal{H}^1.$$

Pohozaev identity

<u>In d dimensions</u>

$$\operatorname{div}\left((x\cdot\nabla\psi)\,\nabla\psi-\frac{1}{2}|\nabla\psi|^2\,x\right)+\frac{d-2}{2}\,|\nabla\psi|^2=(x\cdot\nabla\psi)\,\Delta\psi.$$

PDE identities

Any solution to (O) satisfies the following identities:

Pohozaev

$$\int_{\mathcal{S}} (x \cdot n)(\partial_n \psi)^2 \, \mathrm{d}\mathcal{H}^1 = 2\pi$$

Far field

$$\int_{\mathcal{S}} \partial_n \psi \, \mathrm{d}\mathcal{H}^1 = 2\pi$$

Geometric identities

Any solution to (O) satisfies the following identities:

Gauß–Bonnet

$$\int_{\mathcal{S}} H \,\mathrm{d}\mathcal{H}^1 = 2\pi$$

Minkowski

$$\int_{\mathcal{S}} x \cdot n \,\mathrm{d}\mathcal{H}^1 = 2\pi, \qquad \int_{\mathcal{S}} H(x \cdot n) \,\mathrm{d}\mathcal{H}^1 = \mathcal{H}^1(\mathcal{S})$$

Combined identities

We plug this in the earlier identities:

multiplied by 1

$$-\frac{\mathrm{We}}{2}\int_{\mathcal{S}}(\partial_n\psi)^2\,\mathrm{d}\mathcal{H}^1+2\pi=\lambda\mathcal{H}^1(\mathcal{S})$$

multiplied by $(x\cdot n)$

$$-\mathrm{We}\pi+\mathcal{H}^1(\mathcal{S})=2\pi\lambda$$

Cauchy–Schwarz conclusion

By the Cauchy–Schwarz inequality and eliminating λ we conclude

$$\begin{aligned} 0 &\leq \int_{\mathcal{S}} (\partial_n \psi)^2 \,\mathrm{d}\mathcal{H}^1 - \frac{4\pi^2}{\mathcal{H}^1(\mathcal{S})} \\ &= \frac{1}{\mathrm{We}\pi\mathcal{H}^1(\mathcal{S})}(\mathcal{H}^1(\mathcal{S}) - 2\pi)(\mathcal{H}^1(\mathcal{S}) + 2\pi)(\pi\mathrm{We} - \mathcal{H}^1(\mathcal{S})). \end{aligned}$$

(recall $\left|\mathcal{B}^{\mathrm{in}}\right| = \pi$)

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Video references

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