Kinetic maximal L^p-regularity

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Moving particles

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Physics/Biology/Economics

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First order differential operator $\partial_t + v \cdot \nabla_x$

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Lebesgue spaces L^p with $p \in (1, \infty)$

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First order differential operator $\partial_t + v \cdot \nabla_x$

optimal regularity estimates

Lebesgue spaces L^p with $p \in (1, \infty)$

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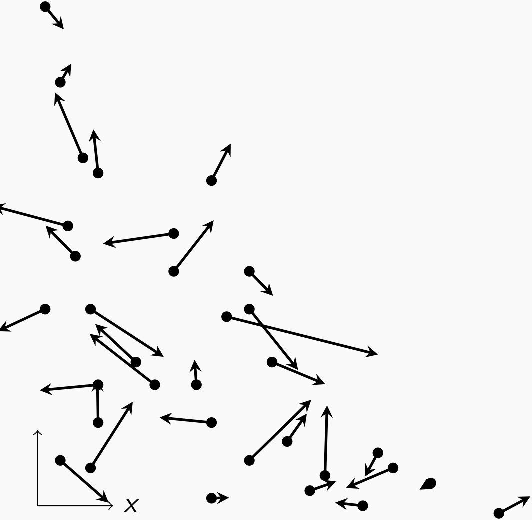
First order differential operator $\partial_t + v \cdot \nabla_x$

Particle physics



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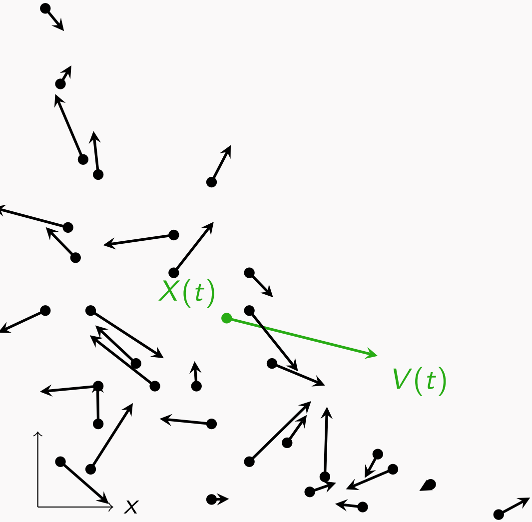

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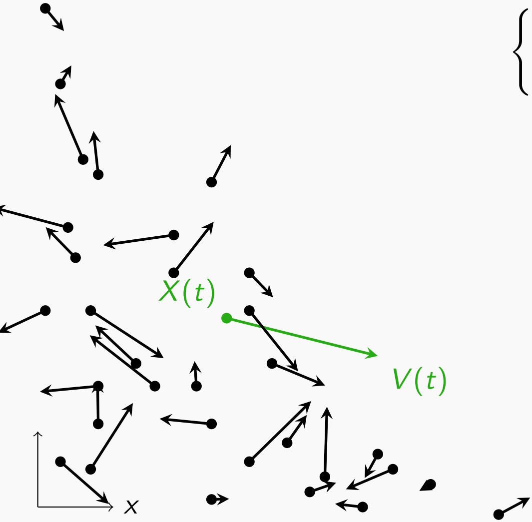
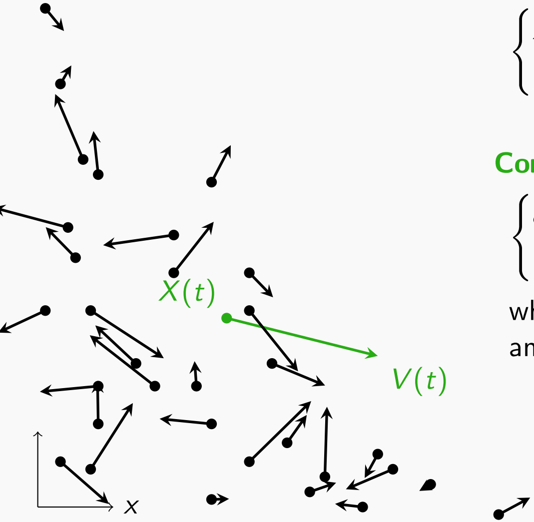


Free transport

$$\begin{cases} X(t) = tV_0 + X_0 \\ V(t) = V_0 \end{cases}$$

Particle physics



Free transport

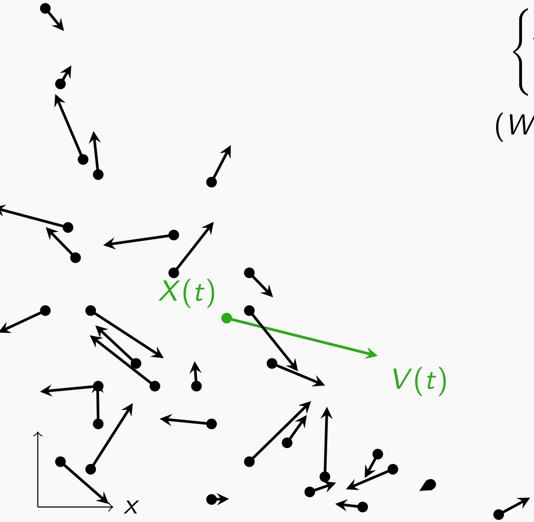
$$\begin{cases} X(t) = tV_0 + X_0 \\ V(t) = V_0 \end{cases}$$

Corresponding PDE

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = 0 \\ u(0) = g \end{cases}$$

where $u = u(t, x, v)$ particle density and $g(x, v)$ initial distribution

Particle physics



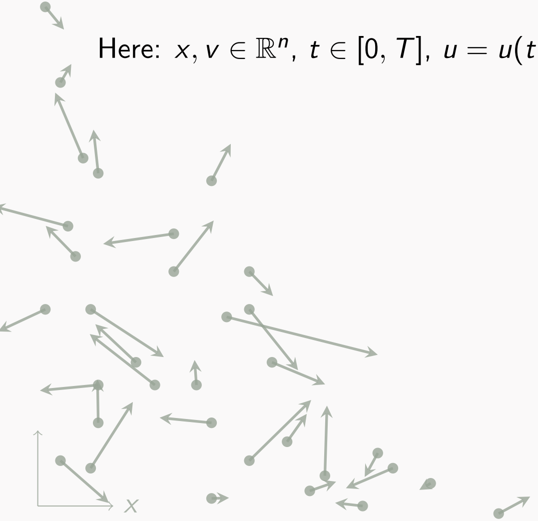
Simple collision model

$$\begin{cases} X(t) = \int_0^t V(s)\mathrm{d}s + X_0 \\ V(t) = W(t) + V_0 \end{cases}$$

$(W(t))_{t\geq 0}$ Wiener process

Kolmogorov equation

Here: $x, v \in \mathbb{R}^n$, $t \in [0, T]$, $u = u(t, x, v)$ particle density

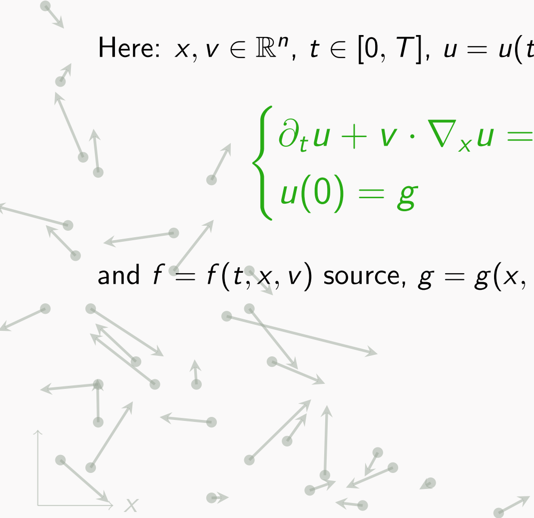


Kolmogorov equation

Here: $x, v \in \mathbb{R}^n$, $t \in [0, T]$, $u = u(t, x, v)$ particle density

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

and $f = f(t, x, v)$ source, $g = g(x, v)$ initial datum.



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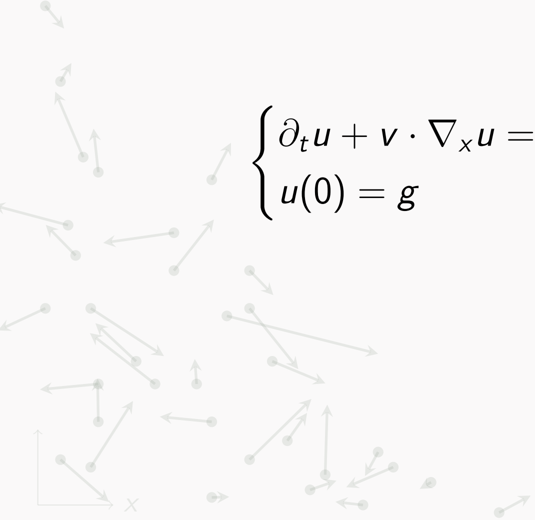

Kolmogorov equation

2nd order PDE, degenerate, unbounded lower order term
reminds of Ornstein-Uhlenbeck equation
hypoelliptic, Hörmander operator

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Kolmogorov equation

2nd order PDE, degenerate, unbounded lower order term
reminds of Ornstein-Uhlenbeck equation
Hörmander operator - hypoelliptic

$$\begin{cases} X_0 u = \sum_{i=1}^{n} X_i^2 u + f \\ u(0) = g \end{cases}$$

where $X_0 = \partial_t + v \cdot \nabla_x$ and $X_i = \partial_{v_i}$

$$[\partial_{v_i}, \partial_t + v \cdot \nabla_x]u = \partial_{v_i}(\partial_t + v \cdot \nabla_x)u - (\partial_t + v \cdot \nabla_x)\, \partial_{v_i} u = \partial_{x_i} u$$

Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Scaling: $\lambda \mapsto (\lambda^2 t, \lambda^3 x, \lambda v)$
Translation: $(t_0, x_0, v_0) \mapsto (t - t_0, x - x_0 - (t - t_0)v, v - v_0)$

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Goal: Determine function spaces X for f, X_γ for g and Z for u such that there exists a unique solution $u \in Z$ of the Kolmogorov equation if and only if $f \in X$ and $g \in X_\gamma$.

Kolmogorov equation

Kinetic maximal regularity

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

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Kolmogorov equation

Kinetic maximal L^p-regularity

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Ansatz:

Kolmogorov equation

Kinetic maximal L^p-regularity

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Ansatz: $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$ with $p \in (1, \infty)$.

Kolmogorov equation

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What is the solution space Z?
What is the trace space X_γ?

Kolmogorov equation

Kinetic maximal L^p-regularity

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What is the solution space Z?

What is the trace space X_γ?

Divide and conquer

Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = 0 \end{cases}$$

Parabolic ansatz - maximal L^p-regularity

$$Z = \{u : \; u, \; \partial_t u, \; \Delta_v u - v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

Kolmogorov equation

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$$\sout{Z = \{u : u, \partial_t u, \Delta_v u - v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}}$$

The desired characterisation fails.

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$$\sout{Z = \{u : u, \partial_t u, \Delta_v u - v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}}$$

The desired characterisation fails.
Indeed: $\sigma(\Delta_v - v \cdot \nabla_x) = \{\lambda \in \mathbb{C} \colon \mathrm{Re}\lambda \leq 0\}$
Metafune 01, Fornaro, Metafune, Pallara & Schnaubelt 22

Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = 0 \end{cases}$$

Solution space

$$Z = \{u \,:\; u,\; \partial_t u + v \cdot \nabla_x u,\; \Delta_v u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}.$$

Kolmogorov equation

Fundamental solution (Kolmogorov 1934):

$$\Gamma(t,x,v) = \frac{c_n}{t^{2n}} \exp\left(-\frac{1}{t}\left|v\right|^2 + \frac{3}{t^2}\langle v, x\rangle - \frac{3}{t^3}\left|x\right|^2\right).$$

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Solution of Kolmogorov equation with $g = 0$ is given by

$$u(t,x,v) = \int_0^t \int_{\mathbb{R}^{2n}} \Gamma(t-s, x-y-(t-s)w, v-w) f(s,y,w) \mathrm{d}(y,w) \mathrm{d}s$$

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Singular integral on homogeneous group (Folland-Stein 1974):

$$\left\|\partial_t u + v\cdot\nabla_x u\right\|_p + \left\|\Delta_v u\right\|_p \leq C\left\|f\right\|_p.$$

For every $f \in L^p((0,T);L^p(\mathbb{R}^{2n}))$ there exists a unique solution $u \in Z$ of the Kolmogorov equation.

Kinetic trace

Temporal trace $u(t)$ is well-defined. In particular

$$\left\{u\colon u, \partial_t u + v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\right\} \hookrightarrow C([0,T]; L^p(\mathbb{R}^{2n}))$$

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The trace space of Z is defined as

$$X_\gamma = \{g\colon \exists u \in Z \text{ with } u(0) = g\}$$

$$\|g\|_{X_\gamma} = \inf_{\substack{u \in Z \\ u(0)=g}} \|u\|_Z.$$

Moreover

$$Z \hookrightarrow C([0,T]; X_\gamma).$$

Kinetic maximal L^p-regularity

Definition

We say that $A\colon D(A) \subset L^p(\mathbb{R}^{2n}) \to L^p(\mathbb{R}^{2n})$ admits kinetic maximal L^p-regularity if for all $f \in X = L^p((0,T);L^p(\mathbb{R}^{2n}))$ there exists a unique distributional solution

$$u \in Z = \{w \;:\; w,\, \partial_t w + v \cdot \nabla_x w,\, Aw \in L^p((0,T);L^p(\mathbb{R}^{2n}))\}$$

of the Cauchy problem

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = Au + f \\ u(0) = 0 \end{cases}$$

In particular $u \in C([0,T];X_\gamma)$.

Kolmogorov equation

Theorem (Folland & Stein 74, Bramanti et al. 10, N. & Zacher 22)

The operator $\Delta_v \colon H^{2,p}_v(\mathbb{R}^{2n}) \to L^p(\mathbb{R}^{2n})$/the Kolmogorov equation admits kinetic maximal $L^p(L^p)$-regularity for all $p \in (1, \infty)$.

Kolmogorov equation

Theorem

For the Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

there exists a unique solution

$$u \in Z = \{w : \ w, \partial_t w + v \cdot \nabla_x w, \Delta_v w \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

if and only if

(i) $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$

(ii) $g \in X_\gamma$.

Moreover, $u \in C([0,T]; X_\gamma)$.

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Kinetic trace

Kinetic Regularisation (Bouchut 02)

Let $u \in L^p(\mathbb{R}^{1+2n})$ with $\partial_t u + v \cdot \nabla_x u \in L^p(\mathbb{R}^{1+2n})$ and $\Delta_v u \in L^p(\mathbb{R}^{1+2n})$. Then

$$D_x^{\frac{2}{3}} u \in L^p(\mathbb{R}^{1+2n}).$$

Here: $D_x^s = (-\Delta_x)^{s/2}$

Kinetic trace

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$$D_x^{\frac{2}{3}} u \in L^p(\mathbb{R}^{1+2n}).$$

Recall the scaling: $\lambda \mapsto (\lambda^2 t, \lambda^3 x, \lambda v)$.

Here: $D_x^s = (-\Delta_x)^{s/2}$

Kinetic trace

Theorem (N. & Zacher 22)

Let $p \in (1, \infty)$ and X_γ the trace space to

$$Z = \{u : \; u, \, \partial_t u + v \cdot \nabla_x u, \, \Delta_v u \in L^p((0, T); L^p(\mathbb{R}^{2n}))\}.$$

Then

$$X_\gamma \cong B_{pp,x}^{\frac{2}{3}(1-\frac{1}{p})}(\mathbb{R}^{2n}) \cap B_{pp,v}^{2(1-\frac{1}{p})}(\mathbb{R}^{2n}).$$

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(ii) $g \in X_\gamma = B_{pp,x}^{\frac{2}{3}(1-\frac{1}{p})}(\mathbb{R}^{2n}) \cap B_{pp,v}^{2(1-\frac{1}{p})}(\mathbb{R}^{2n})$.

Moreover, $u \in C([0,T]; X_\gamma)$.

Extensions

- fractional Kolmogorov equation
- temporal weights
- different base spaces
- variable coefficients

Fractional Kolmogorov equation

with $\beta \in (0,2)$:

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\beta/2} u + f \\ u(0) = g. \end{cases}$$

Fractional Kolmogorov equation

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Theorem (Chen & Zhang 18; Huang, Menozzi & Priola 19)

For $\beta \in (0,2)$ the operator $-(-\Delta_v)^{\beta/2} \colon H_v^{\beta,p}(\mathbb{R}^{2n}) \to L^p(\mathbb{R}^{2n})$ admits kinetic maximal $L^p(L^p)$-regularity for all $p \in (1,\infty)$.

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Theorem (N. & Zacher 22)

$$X_\gamma \cong B_{pp,x}^{\frac{\beta}{\beta+1}(1-\frac{1}{p})}(\mathbb{R}^{2n}) \cap B_{pp,v}^{\beta(1-\frac{1}{p})}(\mathbb{R}^{2n})$$

Temporal weights

Replace $L^p((0,T);X)$ with

$$L^p_\mu((0,T);X) = \left\{u\colon t^{1-\mu}u \in L^p((0,T);X)\right\}$$

with $\mu \in (1/p, 1]$ (Muckenhoupt weight, Prüss & Simonett 04).

Temporal weights

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Advantages:

- Theorem (N. & Zacher 22):
  Kinetic maximal L^p_μ-regularity is independent of $\mu \in (1/p, 1]$.
- for (fractional) Kolmogorov equation:

$$X_{\gamma,\mu} \cong B^{\frac{\beta}{\beta+1}(\mu-\frac{1}{p})}_{pp,x}(\mathbb{R}^{2n}) \cap B^{\beta(\mu-\frac{1}{p})}_{pp,v}(\mathbb{R}^{2n})$$

- treat lower initial value regularity
- allows to observe instantaneous regularisation

Different base spaces

- Kinetic maximal $L^p(L^q)$-regularity for $-(-\Delta_v)^{\beta/2}$ (& variants) with $p, q \in (1, \infty)$

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- Kinetic maximal $L^p(L^q_{j,k})$-regularity for $-(-\Delta_v)^{\beta/2}$ (& variants) with $p, q \in (1, \infty)$ and $j, k \in \mathbb{R}$ where $L^q_{j,k}$ is weighted with $(1 + |v|)^j$ and $(1 + |x| + |v|)^k$

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- Kinetic maximal $L^p(X^s_\beta)$-regularity for $-(-\Delta_v)^{\beta/2}$ with $p \in (1, \infty)$ and $s \geq -1/2$

$$X^s_\beta = \left\{ f \in \mathcal{S}' : \ \left(1 + |\xi|^\beta + |k|^{\frac{\beta}{\beta+1}}\right)^s \mathcal{F}(f) \in L^2 \right\}$$

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Including a characterisation of the trace space.

Kolmogorov equation with variable coefficients

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = a(t, x, v) \colon \nabla_v^2 u + f \\ u(0) = g \end{cases}$$

Under which assumptions on the coefficient $a(t, x, v)$
do we obtain kinetic maximal L^p-regularity?

Kolmogorov equation with variable coefficients

Theorem (Bramanti et al. 13, N. & Zacher 22)

Let $a = a(t, x, v) \in L^\infty([0, T] \times \mathbb{R}^{2n}; \mathrm{Sym}(n))$ with $\lambda |\xi|^2 \le \langle a(t, x, v)\xi, \xi\rangle$ for all (t, x, v) and $\xi \in \mathbb{R}^n$.

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$\forall \varepsilon > 0 \colon \exists \delta > 0$ such that $|t - s| + |x - y - (t - s)v| + |v - w| < \delta$ implies $|a(t, x, v) - a(s, y, w)| < \varepsilon$

OR

$\forall \varepsilon > 0 \colon \exists \delta > 0$ such that $|t - s| + |x - y| + |v - w| < \delta$ implies $|a(t, x, v) - a(s, y, w)| < \varepsilon$.

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Kolmogorov equation with variable coefficients

Theorem (Bramanti et al. 13, N. & Zacher 22)

Let $a = a(t, x, v) \in L^\infty([0, T] \times \mathbb{R}^{2n}; \mathrm{Sym}(n))$ with $\lambda |\xi|^2 \leq \langle a(t, x, v)\xi, \xi\rangle$ for all (t, x, v) and $\xi \in \mathbb{R}^n$. Suppose

$\forall \varepsilon > 0$: $\exists \delta > 0$ such that $|t - s| + |x - y - (t - s)v| + |v - w| < \delta$ implies $|a(t, x, v) - a(s, y, w)| < \varepsilon$

OR

$\forall \varepsilon > 0$: $\exists \delta > 0$ such that $|t - s| + |x - y| + |v - w| < \delta$ implies $|a(t, x, v) - a(s, y, w)| < \varepsilon$.

Then the family of operators $A(t) = a(t, x, v) \colon \nabla_v^2 \colon H_v^{2,p} \to L^p$ admits kinetic maximal $L^p_\mu(L^q_{j,k})$-regularity.

Fractional Kolmogorov equation with variable density

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\beta/2} u + f \\ u(0) = g \end{cases}$$

Recall

$$-(-\Delta_v)^{\beta/2} u = c_{n,\beta} \ \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t,x,v+h) - u(t,x,v)}{|h|^{n+\beta}} \mathrm{d}h$$

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Fractional Kolmogorov equation with variable density

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = A(t)u + f \\ u(0) = g \end{cases}$$

Recall

$$A(t)u = \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(t,x,v+h) - u(t,x,v)}{|h|^{n+\beta}} a(t,x,v,h) \mathrm{d}h$$

Fractional Kolmogorov equation with variable density

Theorem (N. 22):

Let $a = a(t, x, v, h) \in L^\infty([0, T] \times \mathbb{R}^{3n})$ symmetric in h with $0 < \lambda \le a \le \Lambda$, $\alpha \in (0, 1)$, $\alpha < \alpha_0 < 1$

$$\sup \frac{|a(t, x, v, h) - a(s, y, w, h)|}{|t - s|^{\alpha_0} + |x - y - (t - s)v|^{\alpha_0} + |v - w|^{\alpha_0}} < \infty.$$

Fractional Kolmogorov equation with variable density

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Then, the family of operators

$$A(t)u = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t,x,v+h) - u(t,x,v)}{|h|^{n+\beta}} a(t,x,v,h) \mathrm{d}h$$

admits kinetic maximal $L^p_\mu(L^p)$-regularity for all $p > \frac{n}{\alpha}$, $\mu \in (1/p, 1]$.

Same trace space as for $-(-\Delta_v)^{\beta/2}$.

Application to quasilinear equations

Application to quasilinear equations

Think of X as $L^q_{j,k}(\mathbb{R}^n)$ and let $D \subset X$. Seek solutions in

$$Z(0,T) = \{u\colon u, \partial_t u + v\cdot\nabla_x u \in L^p_\mu((0,T);X)\} \cap L^p_\mu((0,T);D)$$

of

$$\begin{cases} \partial_t u + v\cdot\nabla_x u = A(u)u + F(u) \\ u(0) = g \end{cases}$$

where
- $g \in X_{\gamma,\mu}$

- $A\colon X_{\gamma,\mu} \to \mathcal{B}(D,X)$

- $F\colon X_{\gamma,\mu} \to X$

Application to quasilinear equations

Theorem (N. & Zacher 22):

Assume that

$-(A, F) \in C^{1-}_{\mathrm{loc}}(X_{\gamma,\mu}; \mathcal{B}(D, X) \times X)$

- $A(g)$ admits kinetic maximal $L^p_\mu(X)$-regularity.

Then, there exists $T = T(g)$ and $\varepsilon = \varepsilon(g) > 0$ such that

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = A(u)u + F(u) \\ u(0) = h \end{cases}$$

admits a unique solution in $Z(0, T)$ for all $h \in \overline{B_\varepsilon(g)}^{X_{\gamma,\mu}}$.
Moreover, solutions depend continuously on the initial datum.

Boltzmann equation

$$\partial_t u + v \cdot \nabla_x u = Q(u, u) + \mathrm{l.o.t.},$$

Boltzmann equation

In Carleman coordinates

$$\partial_t u + v \cdot \nabla_x u = Q(u, u) + \mathrm{l.o.t.},$$

where

$$Q(u, u) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t, x, v + h) - u(t, x, v)}{|h|^{n+2s}} m(u)(t, x, v, h) \mathrm{d}h$$

Boltzmann equation

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with

$$m(u)(t,x,v,h) = \int_{w \perp h} u(t,x,v+w) \, |w|^{\gamma+2s+1} \, \mathrm{d}w$$

and $s \in (0,1)$, $\gamma > -n$ depend on physical assumptions.

Boltzmann equation

In Carleman coordinates

$$\partial_t u + v \cdot \nabla_x u = Q(u, g) + \mathrm{l.o.t.},$$

where

$$Q(u, g) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t, x, v + h) - u(t, x, v)}{|h|^{n+2s}} m(g)(t, x, v, h) \mathrm{d}h$$

with

$$m(g)(t, x, v, h) = \int_{w \perp h} g(t, x, v + w) \, |w|^{\gamma + 2s + 1} \, \mathrm{d}w$$

and $s \in (0, 1)$, $\gamma > -n$ depend on physical assumptions.

Boltzmann equation - linearised

Fix g and consider the linear equation

$$\partial_t u + v \cdot \nabla_x u = Au$$

where

$$Au = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t,x,v+h) - u(t,x,v)}{|h|^{n+2s}} m(g)(t,x,v,h) \mathrm{d}h$$

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Even if g is very nice the density $m(g)$ is degenerate.

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Even if g is very nice the density $m(g)$ is degenerate.

Earlier Theorem is too restrictive for the Boltzmann equation!

Vlasov-Poisson-Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u + E(u) \cdot \nabla_v u = \nu \Delta_v u \\ u(0) = g \end{cases}$$

where

$$E(t,x) = \frac{\theta}{\sigma_{n-1}} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} M(u)(t,y)\mathrm{d}y$$

with $[M(u)](t,x) = \int_{\mathbb{R}^n} u(t,x,v)\mathrm{d}v$, $\theta = \pm 1$ and $\nu > 0$.

Vlasov-Poisson-Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u + E(u) \cdot \nabla_v u = \nu \Delta_v u \\ u(0) = g \end{cases}$$

Theorem (N. & Zacher):

Let $p, q \in (1, \infty)$, $\mu \in (1/p, 1]$ with $\mu - 1/p > 2n/q$, $j > n$ and $k > n$ then for all initial values

$$g \in B^{\frac{2}{3}(1-\frac{1}{p})}_{qp,x,j,k}(\mathbb{R}^{2n}) \cap B^{2(1-\frac{1}{p})}_{qp,v,j,k}(\mathbb{R}^{2n})$$

there exists $T = T(g)$ and $\varepsilon > 0$ such that the VPK eq. admits a unique solution

$$u \in \left\{ u \colon u, \partial_t u + v \cdot \nabla_x u, \Delta_v u \in L^p_\mu((0, T); L^q_{j,k}(\mathbb{R}^{2n})) \right\}$$

for every initial value $h \in \overline{B_\varepsilon(g)}^{X_{\gamma,\mu}}$. Moreover, the solutions depend continuously on the initial value.

Outlook

- relax assumptions for the variable coefficients (local/nonlocal)
- weak L^p-solutions
- boundary value problems
- sum of the operators $\partial_t + v \cdot \nabla_x$ and A (non-commuting)
- Kinetic Fokker-Planck equation, i.e. $A = \Delta_v + v \cdot \nabla_v$
- qualitative study of quasilinear problems such as global existence and large time behavior
- L^p-theory of the Boltzmann equation

Advertisement

De Giorgi-Nash-Moser meets kinetic equations

Harnack inequality for kinetic equations
- first proof (GIMV 18); quantitative (GM 22)
- trajectorial interpretation of Moser's proof (elliptic/parabolic)
- trajectorial proof of a kinetic Poincaré inequality (used in GM 22)

L. N and R. Zacher. *A trajectorial interpretation of Moser's proof of the Harnack inequality*. Preprint. arXiv:2212.07977 (2022).

L. N and R. Zacher. *On a kinetic Poincaré inequality and beyond*. Preprint. arXiv:2212.03199 (2022).

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