Kinetic maximal L^p-regularity

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Moving particles

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Physics/Biology/Economics

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First order differential operator $\partial_t + v \cdot \nabla_x$

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Lebesgue spaces L^p with $p \in (1, \infty)$

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First order differential operator $\partial_t + v \cdot \nabla_x$

optimal regularity estimates

Lebesgue spaces L^p with $p \in (1, \infty)$

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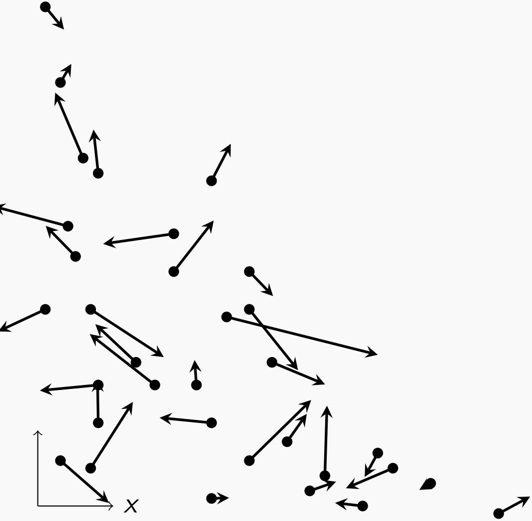
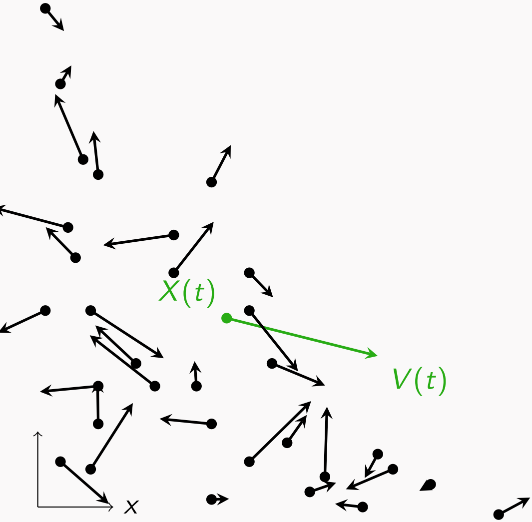
First order differential operator $\partial_t + v \cdot \nabla_x$

Particle physics



Particle physics



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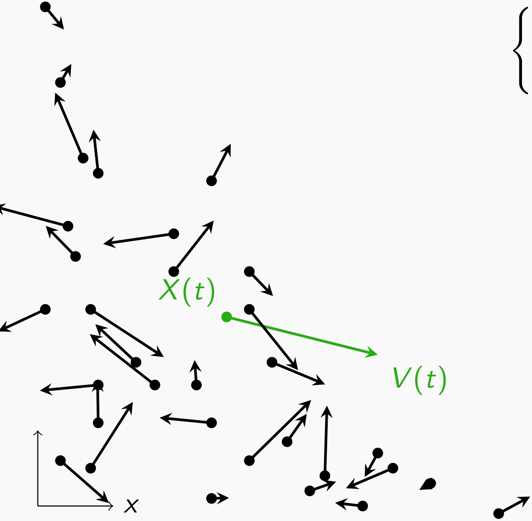
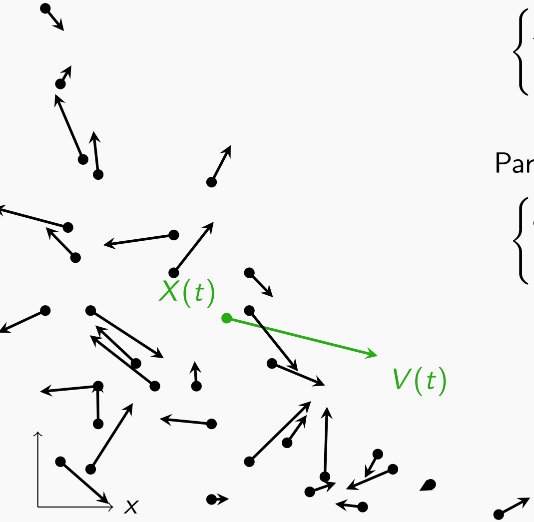


Free transport

$$\begin{cases} X(t) = \int_0^t V(s)\mathrm{d}s + X_0 \\ V(t) = V_0 \end{cases}$$

Particle physics



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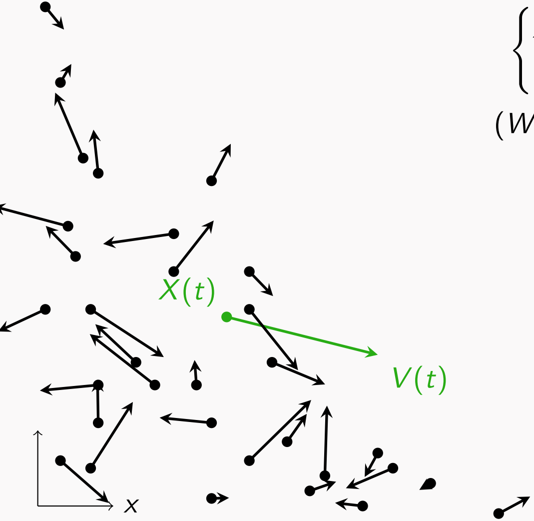
Particle density $u = u(t, x, v)$

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = 0 \\ u(0) = g \end{cases}$$

Particle physics

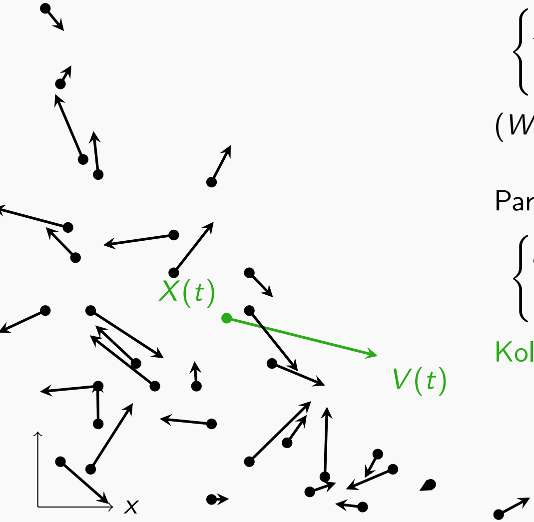


Simple collision model

$$\begin{cases} X(t) = \int_0^t V(s)\mathrm{d}s + X_0 \\ V(t) = W(t) + V_0 \end{cases}$$

$(W(t))_{t\geq 0}$ Wiener process

Particle physics



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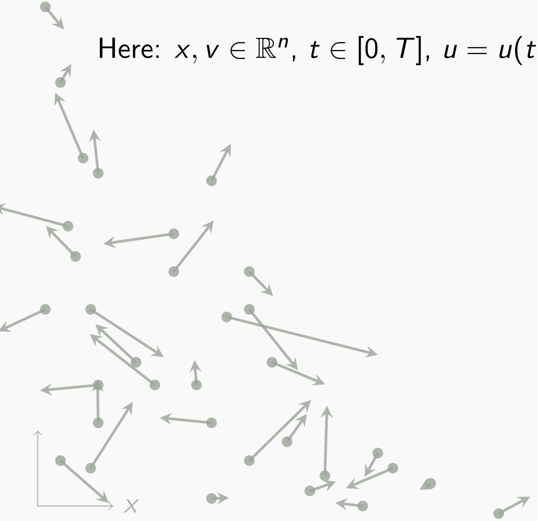
Particle density $u = u(t,x,v)$

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u \\ u(0) = g \end{cases}$$

Kolmogorov equation

Kolmogorov equation

Here: $x, v \in \mathbb{R}^n$, $t \in [0, T]$, $u = u(t, x, v)$ particle density



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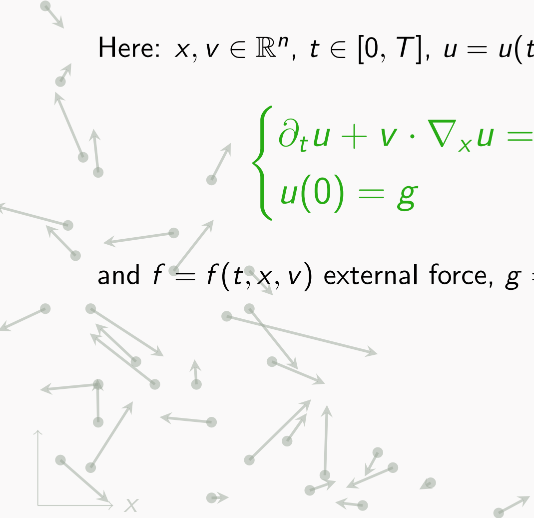
$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

and $f = f(t, x, v)$ external force, $g = g(x, v)$ initial datum.



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Boltzmann/Landau equation

Vlasov-Poisson/Maxwell equation

Kolmogorov equation

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Goal: Determine function spaces X for f, X_γ for g and Z for u such that there exists a unique solution $u \in Z$ of the Kolmogorov equation if and only if $f \in X$ and $g \in X_\gamma$.

Kolmogorov equation

Kinetic maximal regularity

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

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Kolmogorov equation

Kinetic maximal L^p-regularity

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Ansatz:

Kolmogorov equation

Kinetic maximal L^p-regularity

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Ansatz: $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$ with $p \in (1, \infty)$.

Kolmogorov equation

Kinetic maximal L^p-regularity

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Ansatz: $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$ with $p \in (1, \infty)$.

What is the solution space Z?
What is the trace space X_γ?

Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Parabolic theory

$$Z = \{u \,:\; u,\; \partial_t u,\; \Delta_v u - v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Parabolic theory

$$\sout{Z = \{u : u, \partial_t u, \Delta_v u - v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}}$$

The desired characterization fails even for $g = 0$.

Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

Solution space

$$Z = \{u \,:\; u,\ \partial_t u + v \cdot \nabla_x u,\ \Delta_v u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}.$$

Kolmogorov equation

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Solution space

$$Z = \{u : \; u, \; \partial_t u + v \cdot \nabla_x u, \; \Delta_v u \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}.$$

This also sets X_γ.

Kolmogorov equation

Theorem (N. & Zacher, 2022)

Kinetic maximal L^p-regularity of Δ_v

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

There exists a unique solution

$$u \in Z = \{w : \; w, \, \partial_t w + v \cdot \nabla_x w, \, \Delta_v w \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

of the Kolmogorov equation if and only if

(i) $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$

(ii) $g \in X_\gamma = B_{pp,x}^{\frac{2}{3}(1-\frac{1}{p})}(\mathbb{R}^{2n}) \cap B_{pp,v}^{2(1-\frac{1}{p})}(\mathbb{R}^{2n})$.

Moreover, $u \in C([0,T]; X_\gamma)$.

Kolmogorov equation

Theorem (N. & Zacher, 2022)

Kinetic maximal L^p-regularity of Δ_v

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Moreover, $u \in C([0,T]; X_\gamma)$.

Kinetic maximal L^p-regularity

Definition

We say that A admits kinetic maximal L^p-regularity if

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = Au + f \\ u(0) = g \end{cases}$$

there exists a unique solution

$$u \in Z = \{w \,:\; w,\, \partial_t w + v \cdot \nabla_x w,\, Aw \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

of the Cauchy problem if and only if

(i) $f \in X = L^p((0,T); L^p(\mathbb{R}^{2n}))$

(ii) $g \in X_\gamma$.

Moreover, $u \in C([0,T]; X_\gamma)$.

Application

Short time existence for quasilinear kinetic PDEs

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = A(u)u + F(u) \\ u(0) = g \end{cases}$$

if

- $A(g)$ admits kinetic maximal L^p-regularity,

- the nonlinearities A and F are locally Lipschitz.

Boltzmann equation

$$\partial_t u + v \cdot \nabla_x u = Q(u, u) + \mathrm{l.o.t.},$$

Boltzmann equation

In Carleman coordinates

$$\partial_t u + v \cdot \nabla_x u = Q(u,u) + \mathrm{l.o.t.},$$

where

$$Q(u,u) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t,x,v+h) - u(t,x,v)}{|h|^{n+2s}} m(u)(t,x,v,h) \mathrm{d}h$$

Boltzmann equation

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with

$$m(u)(t,x,v,h) = \int_{w \perp h} u(t,x,v+w) \, |w|^{\gamma+2s+1} \, \mathrm{d}w$$

and $s \in (0,1)$, $\gamma > -n$ depend on physical assumptions.

Boltzmann equation

In Carleman coordinates

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and $s \in (0, 1)$, $\gamma > -n$ depend on physical assumptions.

Boltzmann equation - linearised

Consider the linear equation

$$\partial_t u + v \cdot \nabla_x u = Au$$

where

$$Au = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t,x,v+h) - u(t,x,v)}{|h|^{n+2s}} a(t,x,v,h) \mathrm{d}h$$

with

$$0 < \lambda \le a(t,x,v,h) \le \Lambda$$

and $a(t,x,v,h)$ satisfies a continuity property.

Fractional Kolmogorov equation

Theorem (N., 2022):

The operator

$$Au = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t,x,v+h) - u(t,x,v)}{|h|^{n+2s}} a(t,x,v,h) \mathrm{d}h$$

for suitable $a = a(t,x,v,h)$ admits kinetic maximal L^p-regularity.
The trace space can be characterized by

$$X_\gamma \cong B_{pp,x}^{\frac{2s}{2s+1}(1-\frac{1}{p})}(\mathbb{R}^{2n}) \cap B_{pp,v}^{2s(1-\frac{1}{p})}(\mathbb{R}^{2n}).$$

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Too restrictive for the Boltzmann equation!

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