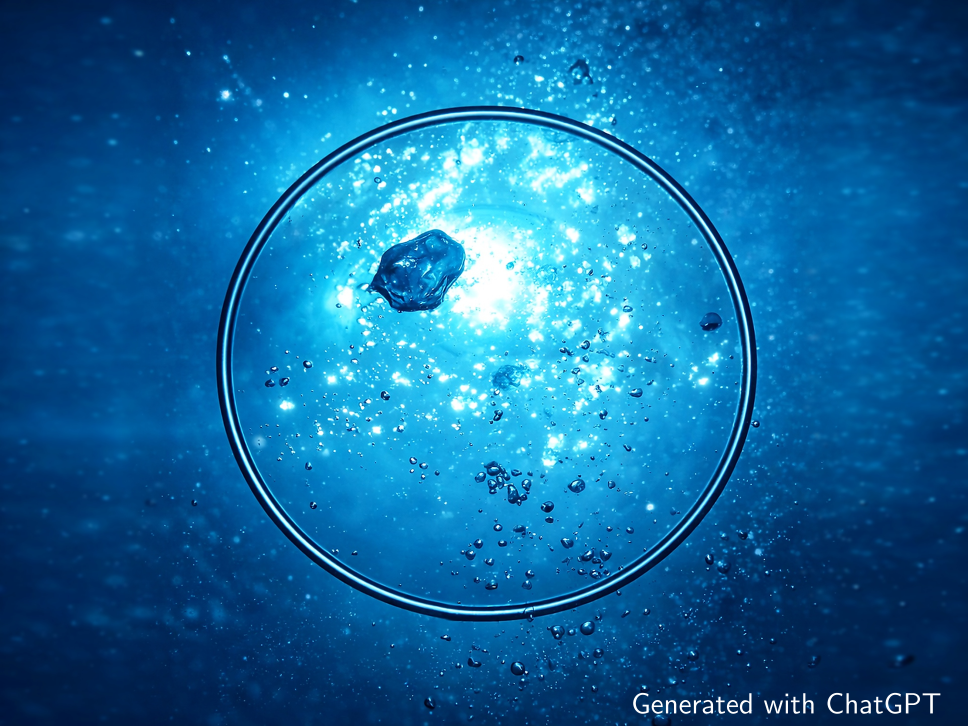
MM Mathematics Münster
Cluster of Excellence

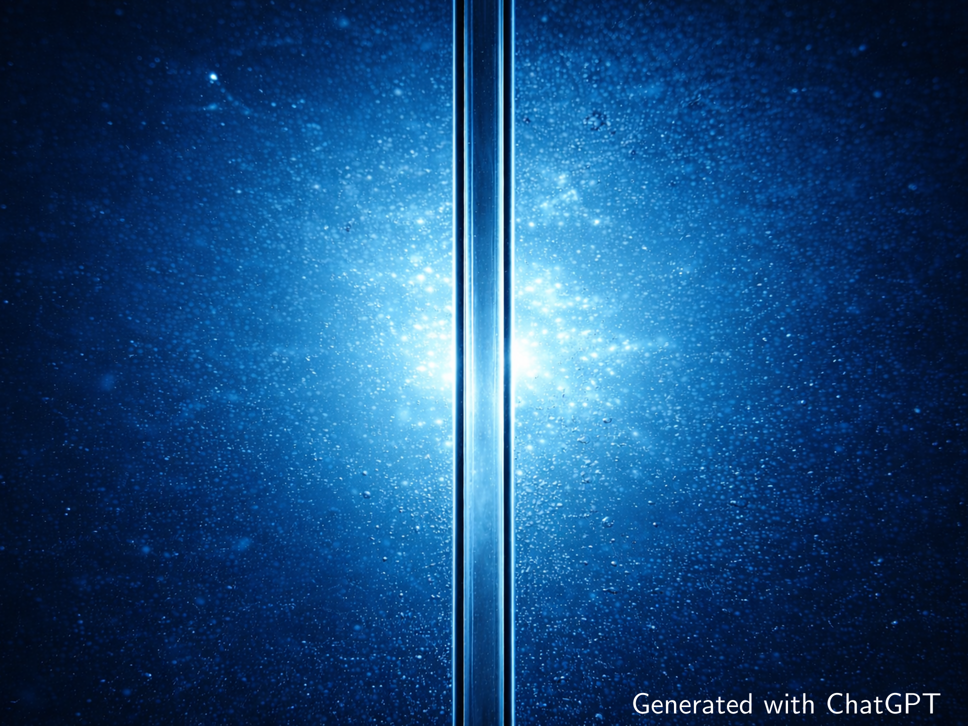
Global rigidity of two-dimensional bubbles

Lukas Niebel
University of Münster

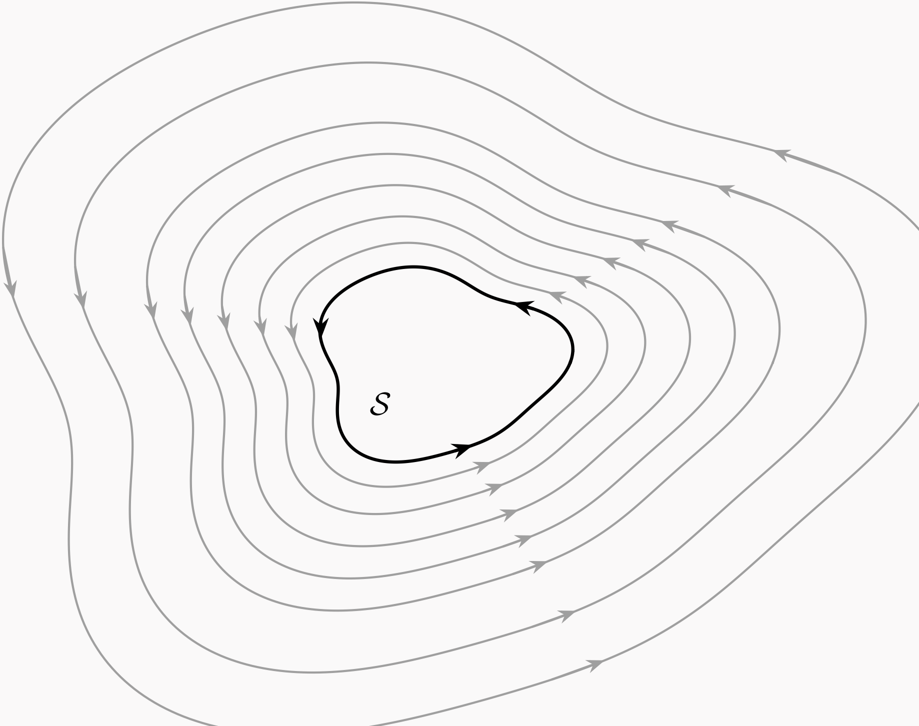
Spring school 2026: Horizons in nonlinear PDEs

Ulm University, 12th March 2026


Generated with ChatGPT



Generated with ChatGPT


$\mathcal{S}$

Free boundary 2D Euler equations

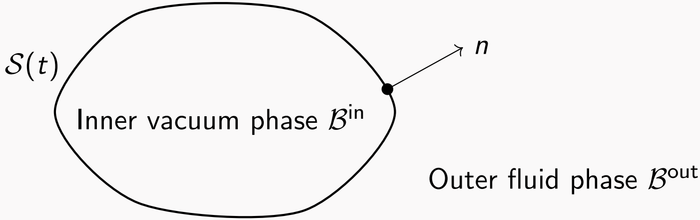
Velocity field of the fluid $U(t,\cdot)\colon \mathcal{B}^{\text{out}}(t) \to \mathbb{R}^2$ solution to

$$\begin{aligned} \partial_t U + (U\cdot\nabla)U + \nabla P &= 0 \qquad && x \in \mathcal{B}^{\text{out}}(t) \\ \nabla \cdot U &= 0 && x \in \mathcal{B}^{\text{out}}(t) \\ U \cdot n &= V_n && x \in \mathcal{S}(t) \end{aligned}$$

for $t \in \mathbb{R}$ and where

- P is the pressure, V_n is the normal speed of the interface,

- $\mathcal{S}(t)$ is the interface separating $\mathcal{B}^{\text{in}}(t)$ and $\mathcal{B}^{\text{out}}(t)$.



Free boundary 2D Euler equations with surface tension

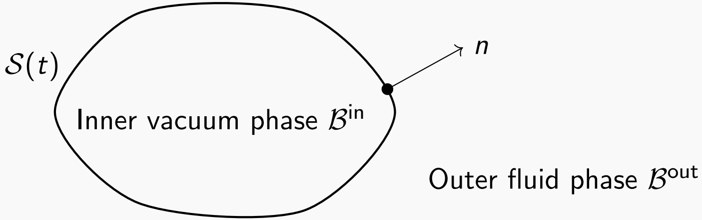
Velocity field of the fluid $U(t,\cdot)\colon \mathcal{B}^{\text{out}}(t) \to \mathbb{R}^2$ solution to

$$\begin{aligned} \partial_t U + (U\cdot\nabla)U + \nabla P &= 0 \qquad && x \in \mathcal{B}^{\text{out}}(t) \\ \nabla \cdot U &= 0 \qquad && x \in \mathcal{B}^{\text{out}}(t) \\ U \cdot n &= V_n \qquad && x \in \mathcal{S}(t) \\ -P &= \sigma H \qquad && x \in \mathcal{S}(t) \end{aligned}$$

for $t \in \mathbb{R}$ and where

- H is the curvature of $\mathcal{S}$

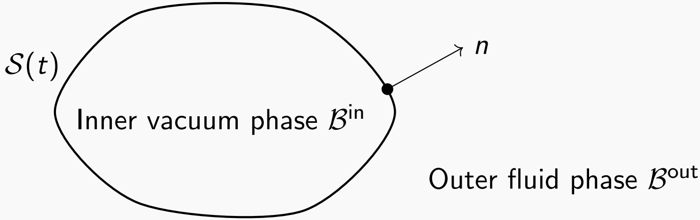
- $\sigma > 0$ is the surface tension



Stationary solutions

Find $U\colon \mathcal{B}^{\text{out}} \to \mathbb{R}^2$ solution to

$$\begin{aligned} (U\cdot\nabla)U + \nabla P &= 0 && \text{in } \mathcal{B}^{\text{out}} \\ \nabla\cdot U &= 0 && \text{in } \mathcal{B}^{\text{out}} \\ U\cdot n &= 0 && \text{on } \mathcal{S} \\ -P &= \sigma H && \text{on } \mathcal{S} \end{aligned}$$



Stationary solutions

Find $U\colon \mathcal{B}^{\text{out}} \to \mathbb{R}^2$ solution to

$$\begin{aligned} (U\cdot\nabla)U + \nabla P &= 0 && \text{in } \mathcal{B}^{\text{out}} \\ \nabla\cdot U &= 0 && \text{in } \mathcal{B}^{\text{out}} \\ U\cdot n &= 0 && \text{on } \mathcal{S} \\ -P &= \sigma H && \text{on } \mathcal{S} \end{aligned}$$

Irrotational flow in the fluid phase $\mathcal{B}^{\text{out}}$

Stream function $U = \nabla^{\perp}\psi$

Rescaling

Weber number $\mathrm{We} \sim \dfrac{1}{\sigma}$

Overdetermined free boundary value problem

Find $\psi\colon \mathcal{B}^{\text{out}} \to \mathbb{R}$ such that

$$
\begin{aligned}
-\Delta\psi &= 0 && \text{in } \mathcal{B}^{\text{out}} \\
\psi &= C_0 && \text{on } \mathcal{S} \\
\psi(x) &= \log|x| + O(1) && \text{as } |x| \to \infty \\
-\frac{1}{2}\mathrm{We}\,(\partial_n\psi)^2 + H &= \lambda && \text{on } \mathcal{S}
\end{aligned}
\qquad \text{(O)}
$$

with

- Weber number $\mathrm{We} \geq 0$
- constants $C_0, \lambda \in \mathbb{R}$
- closed Jordan curve $\mathcal{S} \in \mathrm{C}^{1,1}$
- $\mathcal{S}$ partitions $\mathbb{R}^2$ into $\mathcal{B}^{\text{in}}$ and $\mathcal{B}^{\text{out}}$ with $\left|\mathcal{B}^{\text{in}}\right| = \pi$

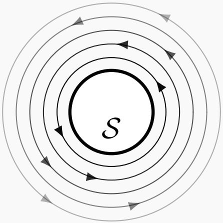
Circular solution

- $\mathcal{S} = \partial B_1(0)$

- $H = 1$

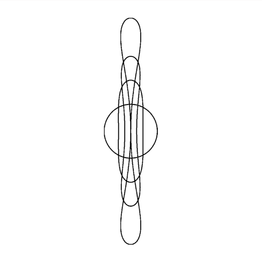
- $\psi = C_0 + \log|x|$, hence $\partial_n \psi = 1$ on $\mathcal{S}$

- $\lambda = -\frac{1}{2}\mathrm{We} + 1$

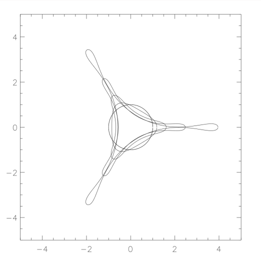


Non-circular solutions

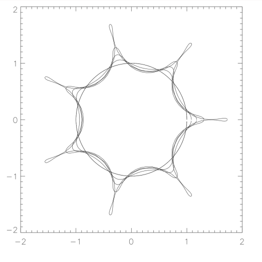
exist for all $\mathrm{We} \in \{3, 4, 5, 6, \dots\}$.



$\mathrm{We} = 3$ [1]



$\mathrm{We} = 4$ [2]



$\mathrm{We} = 8$ [2]

[1] D. Crowdy, *Circulation-induced shape deformations of drops and bubbles: Exact two-dimensional models*, Phys. Fluids **11** 2836 (1999), 2836–2845.

[2] R. Wegmann and D. Crowdy, *Shapes of two-dimensional bubbles deformed by circulation*, Nonlinearity **13** (2000), 2131–2141.

Global rigidity

Theorem (N., 2025)

Let $(\mathcal{S}, \psi, C_0, \mathrm{We}, \lambda)$ be a solution to (O), which is not a circle, then $\mathrm{We} > 2$.

Proof

Integrate the jump condition $-\frac{1}{2}\mathrm{We}\,(\partial_n\psi)^2 + H = \lambda$ on $\mathcal{S}$

multiplied by 1

$$-\frac{\mathrm{We}}{2}\int_{\mathcal{S}}(\partial_n\psi)^2\,\mathrm{d}\mathcal{H}^1 + \int_{\mathcal{S}} H\,\mathrm{d}\mathcal{H}^1 = \lambda\mathcal{H}^1(\mathcal{S})$$

multiplied by $(x\cdot n)$

$$-\frac{\mathrm{We}}{2}\int_{\mathcal{S}}(x\cdot n)(\partial_n\psi)^2\,\mathrm{d}\mathcal{H}^1 + \int_{\mathcal{S}} H(x\cdot n)\,\mathrm{d}\mathcal{H}^1 = \lambda\int_{\mathcal{S}} x\cdot n\,\mathrm{d}\mathcal{H}^1.$$

Proof

Pohozaev

$$\operatorname{div}\left((x \cdot \nabla\psi)\,\nabla\psi - \frac{1}{2}|\nabla\psi|^2\,x \right) + \frac{d-2}{2}\,|\nabla\psi|^2 = (x \cdot \nabla\psi)\,\Delta\psi.$$

Proof

Any solution to (O) satisfies the following identities:

Pohozaev

$$\int_{\mathcal{S}} (x \cdot n)(\partial_n \psi)^2 \, \mathrm{d}\mathcal{H}^1 = 2\pi$$

Far field

$$\int_{\mathcal{S}} \partial_n \psi \, \mathrm{d}\mathcal{H}^1 = 2\pi$$

Proof

Any solution to (O) satisfies the following identities:

Gauß–Bonnet

$$\int_{\mathcal{S}} H \,\mathrm{d}\mathcal{H}^1 = 2\pi$$

Minkowski

$$\int_{\mathcal{S}} x \cdot n \,\mathrm{d}\mathcal{H}^1 = 2\pi, \qquad \int_{\mathcal{S}} H(x \cdot n) \,\mathrm{d}\mathcal{H}^1 = \mathcal{H}^1(\mathcal{S})$$

Proof

We plug this in the earlier identities:

multiplied by 1

$$-\frac{\mathrm{We}}{2}\int_{\mathcal{S}}(\partial_n\psi)^2\,\mathrm{d}\mathcal{H}^1+2\pi=\lambda\mathcal{H}^1(\mathcal{S})$$

multiplied by $(x\cdot n)$

$$-\mathrm{We}\pi+\mathcal{H}^1(\mathcal{S})=2\pi\lambda$$

Proof

By the Cauchy–Schwarz inequality and eliminating λ we conclude

$$
\begin{aligned}
0 &\le \int_{\mathcal{S}} (\partial_n \psi)^2 \, \mathrm{d}\mathcal{H}^1 - \frac{4\pi^2}{\mathcal{H}^1(\mathcal{S})} \\
&= \frac{1}{\mathrm{We}\pi\mathcal{H}^1(\mathcal{S})}(\mathcal{H}^1(\mathcal{S}) - 2\pi)(\mathcal{H}^1(\mathcal{S}) + 2\pi)(\pi\mathrm{We} - \mathcal{H}^1(\mathcal{S})).
\end{aligned}
$$

References

L. Niebel. *Global rigidity of two-dimensional bubbles*, arXiv:2510.17557 (2025).

Y. Han, and C. Seis. *Non-existence of thick bubble rings at low Weber numbers*, in preparation (2026).

lukasniebel.github.io