# Kinetic maximal $L^p$-regularity and more

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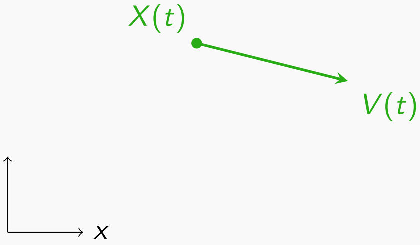
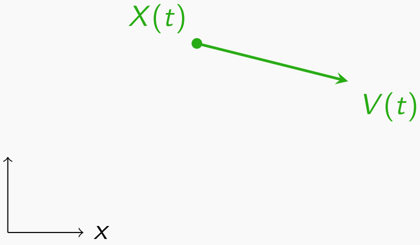
# Particle physics



# Particle physics

**Free transport**

$$\begin{cases} X(t) = tV_0 + X_0 \\ V(t) = V_0 \end{cases}$$



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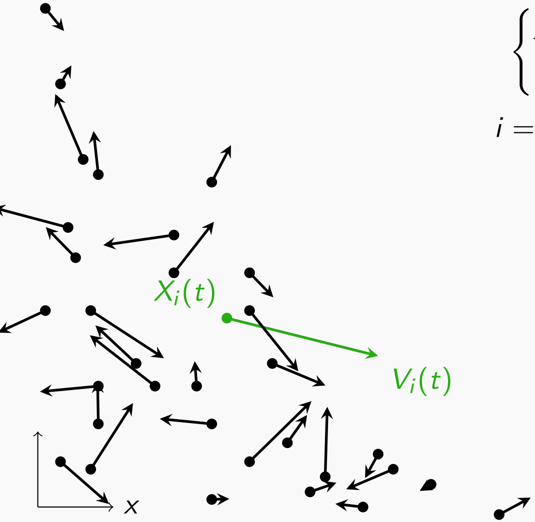
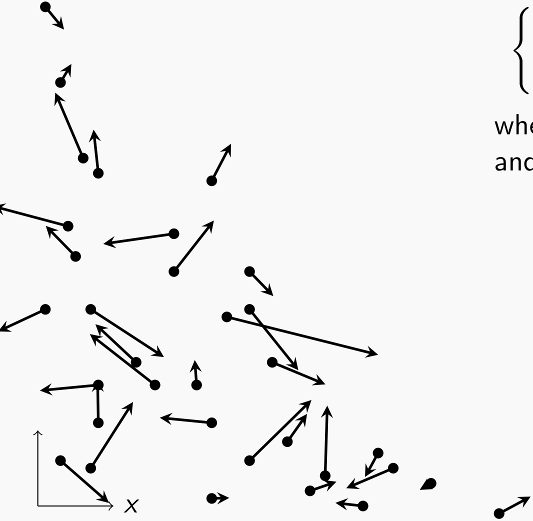



**Free transport**

$$\begin{cases} X_i(t) = tV_{0,i} + X_{0,i} \\ V_i(t) = V_{0,i} \end{cases}$$

$i = 1, \ldots, 10^{23}$

# Particle physics




**Free transport PDE**

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = 0 \\ u(0) = g \end{cases}$$

where $u = u(t, x, v)$ particle density and $g(x, v)$ initial distribution.

# Kolmogorov equation

Here: $x, v \in \mathbb{R}^n$, $t \in [0, T]$, $u = u(t, x, v)$ particle density

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

and $f = f(t, x, v)$ source, $g = g(x, v)$ initial datum.

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Hypoelliptic

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Hypoelliptic
Boltzmann/Landau equation
Vlasov-Poisson/Maxwell equation

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# Kinetic maximal $L^p$-regularity

Theorem (N. & Zacher 22)

For the Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

there exists a unique solution

$$u \in \{w : \ w, \partial_t w + v \cdot \nabla_x w, \Delta_v w \in L^p((0,T); L^p(\mathbb{R}^{2n}))\}$$

if and only if

(i) $f \in L^p((0,T); L^p(\mathbb{R}^{2n}))$

(ii) $g \in X_\gamma = B_{pp,x}^{\frac{2}{3}(1-\frac{1}{p})}(\mathbb{R}^{2n}) \cap B_{pp,v}^{2(1-\frac{1}{p})}(\mathbb{R}^{2n})$.

Moreover, $u \in C([0,T]; X_\gamma)$.

# References


L. N. and R. Zacher, *Kinetic maximal $L^2$-regularity for the (fractional) Kolmogorov equation*. Journal of Evolution Equations 21 (2021).

- $L^2$-estimates, observations on kinetic regularisation

L. N. and R. Zacher, *Kinetic maximal $L^p$-regularity with temporal weights and application to quasilinear kinetic diffusion equations*. Journal of Differential Equations 307 (2022).

- general framework, kinetic trace theorem, Kolmogorov equation with variable coefficients, quasilinear equations

L. N., Kinetic maximal $L^p_\mu(L^p)$-regularity for the fractional Kolmogorov equation with variable density. Nonlinear Analysis (2022).

- replace $\Delta_v$ with $-(-\Delta_v)^{\beta/2}$ and consider a variable density

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# De Giorgi-Nash-Moser theory

Goal: a priori estimates for weak solutions to

$$-\nabla \cdot (A\nabla u) = 0 \qquad \text{elliptic}$$

$$\partial_t u = \nabla \cdot (A\nabla u) \qquad \text{parabolic}$$

where $A = A(x), A(t,x)$ is measurable and

$$\lambda\,|\xi|^2 \le \langle A\xi, \xi\rangle \le \Lambda\,|\xi|^2$$

for $0 < \lambda < \Lambda$.

$L^p$–$L^\infty$ estimates, Harnack inequality, Hölder estimates

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$$\partial_t u + v \cdot \nabla_x u = \nabla_v \cdot (A \nabla_v u) \qquad \text{kinetic}$$

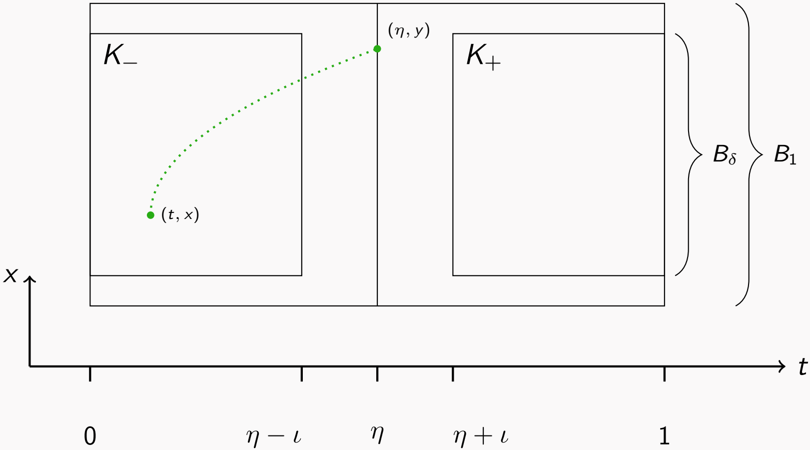
where $A = A(x), A(t,x), A(t,x,v)$ is measurable and

$$\lambda |\xi|^2 \leq \langle A\xi, \xi \rangle \leq \Lambda |\xi|^2$$

for $0 < \lambda < \Lambda$.

$L^p$–$L^\infty$ estimates, Harnack inequality, Hölder estimates

# Moser meets parabolic trajectories




$$\gamma(r) = (t + r^2(\eta - t), x + r(y - x))$$

# Kinetic Poincaré inequality

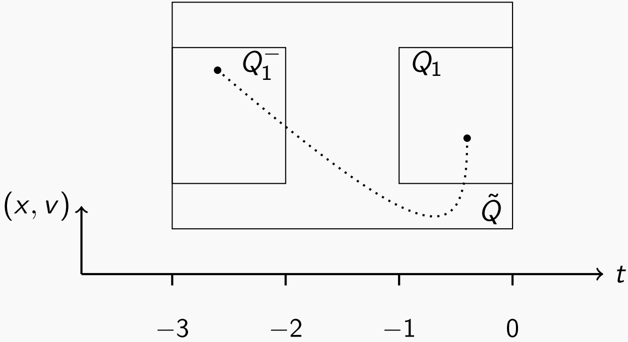
$(1)\ \partial_t u + v \cdot \nabla_x = \nabla_v \cdot (A \nabla_v u)$

<u>Theorem (Guerand & Mouhot 22, N. & Zacher 22):</u>

Let $u \geq 0$ be a subsolution to (1) in $\tilde{Q}$. Then

$$\left\| \left( u - \langle u \varphi^2 \rangle_{Q_1^-} \right)_+ \right\|_{L^1(Q_1)} \leq C \left\| \nabla_v u \right\|_{L^1(\tilde{Q})}.$$

with $\varphi^2$ supported in $Q_1^-$.



# References


L. N. and R. Zacher. *A trajectorial interpretation of Moser's proof of the Harnack inequality*. Preprint. arXiv:2212.07977 (2022).


- elliptic and parabolic, trajectorial proof of a weak $L^1$-estimate for the logarithm of supersolutions


L. N. and R. Zacher. *On a kinetic Poincaré inequality and beyond*. Preprint. arXiv:2212.03199 (2022).


- kinetic trajectories, kinetic Poincaré inequality, higher order hypoelliptic equations

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