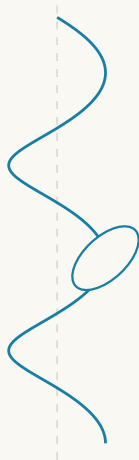


On Leonardo da Vinci's paradox

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ETH Zürich

joint work with Björn Gebhard and Christian Seis



UZH–ETH Seminar: PDE and Mathematical Physics

October 15, 2026

da Vinci at the ETH



Personal photograph



[YouTube Short](#) ■ Spiraling air bubbles

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Wissen > Forscher erklären, warum aufsteigende Blasen torkeln

KLINISCHE FORSCHUNG

**"Leonardos Paradox":
Forscher erklären,
warum aufsteigende
Blasen torkeln**

von Peter Cavallaro | 01. Februar 2018 • 16:01 Uhr • 5 Min.

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WISSENSCHAFT

🔥 Leonardo Paradoxon: 500 Jahre altes Rätsel der Naturwissenschaft gelöst

Warum steigen größere Luftblasen im Wasser nicht auf einer geraden Bahn an die Oberfläche? Ein halbes Jahrtausend, nachdem Leonardo da Vinci diese Frage stellte, haben zwei Physiker eine Erklärung für das Phänomen gefunden.

VON **KATARINA FISCHER**
AKTUALISIERT AM: 27. JANUAR 2023

[link](#)



Leonardo da Vinci, 1452–1519

Image source: National Geographic Deutschland.

da Vinci's spiraling bubbles



Leonardo da Vinci, *Codex Leicester*, fol. 25r (detail), c. 1508–10; reproduced in Prosperetti (2004), Fig. 19.

da Vinci's spiraling bubbles



“The air which is submerged together with the water ... returns to the air, penetrating the water in sinuous movement.”

Leonardo da Vinci
Codex Leicester, fol. 25r

Leonardo da Vinci's paradox

"I have proposed to refer to the failure of bubbles to follow a straight path as Leonardo's paradox."

Andrea Prosperetti, "Bubbles," *Phys. Fluids* **16** (2004), 1852–1865, p. 1861.

Today, we prove the existence of steady-spiralling bubbles in an idealised fluid model.



[YouTube Short](#) ■ Sodium-potassium alloy

A bubble in an ambient liquid

We work in $\mathbb{R}^3$.

$\Omega^{\text{out}}(t)$: exterior liquid

velocity $u(t, x)$

pressure $p(t, x)$

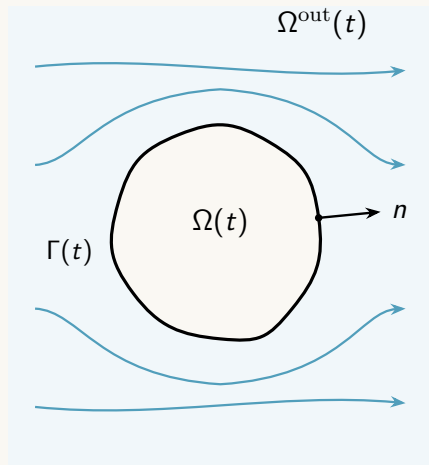
$\Omega(t)$: massless bubble

pressure p_{gas} constant

zero velocity field

Modeling assumptions

1. Inviscid fluid
2. Incompressible fluid
3. Irrotational exterior flow
4. No gravity



Euler equations

$$\rho(\partial_t u + (u \cdot \nabla)u) + \nabla p = 0$$

$$\nabla \cdot u = 0$$

incompressible

$$\nabla \times u = 0$$

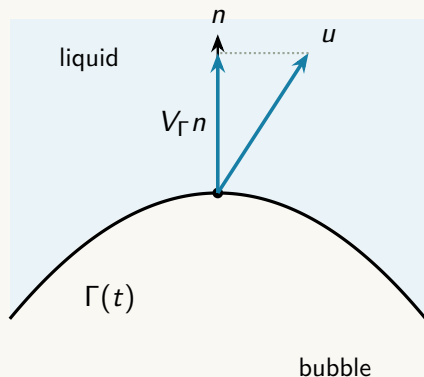
irrotational

Kinematic boundary condition

$$u \cdot n = V_\Gamma$$

The interface and the liquid have the same normal velocity.

Liquid does not cross the free boundary $\Gamma(t)$.



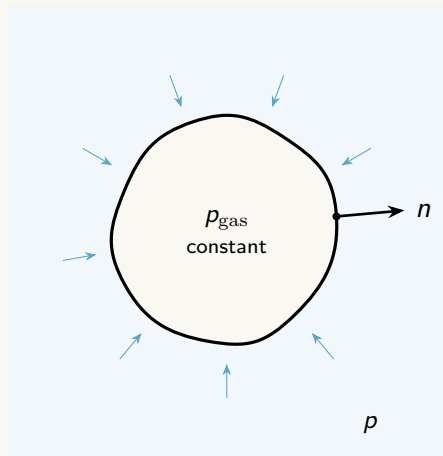
Young–Laplace law

$$p_{\text{gas}} - p = \sigma H$$

$\sigma > 0$: surface-tension coefficient

$$H = \operatorname{div}_{\Gamma} n$$

Sphere of radius R : $H = 2/R$



We seek a spiralling bubble

$\mathcal{R}_\theta$: rotation through angle θ about e_3 .

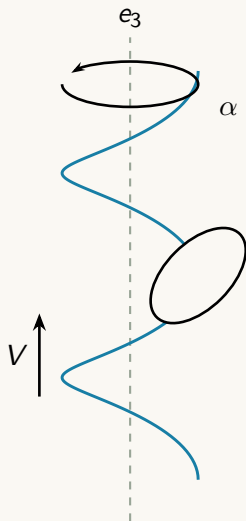
$$x = \mathcal{R}_{\alpha t} y + V t e_3$$

$$\Omega(t) = \mathcal{R}_{\alpha t} \Omega + V t e_3$$

$$u(t, x) = \mathcal{R}_{\alpha t} U(y)$$

V : axial speed; α : angular speed.

The shape and the flow are stationary in the moving frame.



Potential flow

Irrotational flow in a simply connected exterior can be written as

$$u(t, x) = \nabla_x \Phi(t, x)$$

Stationary in the moving frame

$$\Phi(t, x) = \phi(y), \quad y = \mathcal{R}_{-\alpha t}(x - Vte_3)$$

$$\dot{x} = \alpha e_3 \times (\mathcal{R}_{\alpha t} y) + Ve_3 = \mathcal{R}_{\alpha t} \underbrace{(\alpha e_3 \times y + Ve_3)}_{=: W_{\alpha, V}(y)}$$

Bernoulli and Young–Laplace eliminate the pressure:

$$\sigma H - \rho F_{\alpha, V}(y, \nabla \phi) = c$$

$$F_{\alpha, V}(y, q) = \frac{1}{2} |q - Ve_3|^2 + \alpha y \cdot (e_3 \times q)$$

Overdetermined free-boundary value problem

Fix $R, \rho, \sigma > 0$. Find a bounded bubble Ω with simply connected exterior, a potential ϕ , a constant $c \in \mathbb{R}$, and speeds (α, V) such that

$$\left\{ \begin{array}{ll} \Delta \phi = 0 & \text{in } \Omega^{\text{out}}, \\ \partial_n \phi = W_{\alpha, V} \cdot n & \text{on } \Gamma, \\ \sigma H - \rho F_{\alpha, V}(y, \nabla \phi) = c & \text{on } \Gamma, \\ \phi(y) \longrightarrow 0 & \text{as } |y| \longrightarrow \infty, \\ \nabla \phi \in L^2(\Omega^{\text{out}}), \\ |\Omega| = \frac{4\pi}{3} R^3, \\ \int_{\Omega} y_3 \, dy = 0. \end{array} \right.$$

Recall $W_{\alpha, V}(y) = \alpha e_3 \times y + V e_3$ and $F_{\alpha, V}(y, q) = \frac{1}{2} |q - V e_3|^2 + \alpha y \cdot (e_3 \times q)$.

The Weber number

$$\text{We} := \frac{\rho V^2}{\sigma/R} = \frac{\text{inertial pressure}}{\text{capillary pressure}}$$

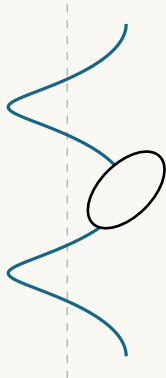
Theorem (Gebhard–N.–Seis '26)

Fix $\rho, \sigma, R > 0$.

For every $0 < We \ll 1$, the capillary Euler equations admit a near-spherical relative equilibrium whose centroid follows a true helix.

The branch is locally unique near the $(2, 1)$ Rayleigh–Lamb frequency, modulo rotations.

$$\alpha \approx \alpha_* = \left(\frac{12\sigma}{\rho R^3} \right)^{1/2}$$



Rigorously confirms predictions by Benjamin (1987) and Meiron (1989).

0. Nondimensionalisation

By rescaling it suffices to consider:

$$R = \rho = \sigma = 1$$

Then

$$\text{We} = V^2 \quad \text{and} \quad \alpha_* = 2\sqrt{3}$$

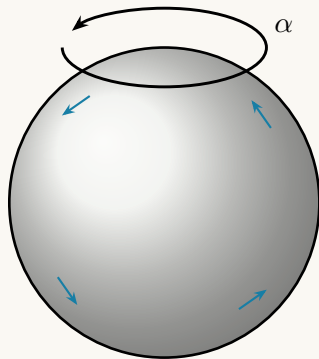
1. The stationary sphere is a solution

$$\Gamma = \mathbb{S}^2, \quad V = 0, \quad \alpha \in \mathbb{R}.$$

$$n = y \implies W_{\alpha,0} \cdot n = 0.$$

$$\partial_n \phi = 0, \quad \phi \rightarrow 0 \implies \phi \equiv 0.$$

$$H - F_{\alpha,0} = 2 = c.$$



blue arrows: $W_{\alpha,0} = \alpha e_3 \times y$,
tangent to the sphere

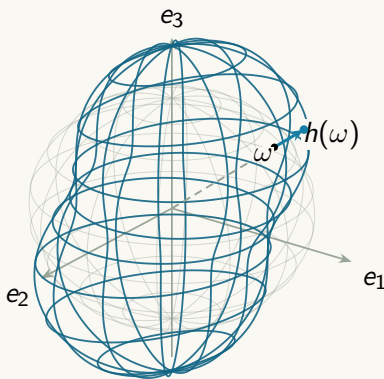
2. Spherical graphs

Write $\omega = (X, Y, Z) \in \mathbb{S}^2$. Let $k \geq 4$.

Take $h \in H^{k+2}(\mathbb{S}^2)$ with $1 + h > 0$.

$$\psi_h(\omega) = (1 + h(\omega))\omega,$$

$$\Gamma_h = \psi_h(\mathbb{S}^2).$$



grey: $\mathbb{S}^2$ blue: Γ_h

2. Spherical graphs

The volume and vertical-centroid constraints are

$$\mathcal{V}(h) := \frac{1}{3} \int_{\mathbb{S}^2} (1+h)^3 \, dS - \frac{4\pi}{3}, \quad \mathcal{C}_3(h) := \frac{1}{4} \int_{\mathbb{S}^2} (1+h)^4 Z \, dS.$$

$$\mathcal{M}^{k+2} := \{h \in H^{k+2}(\mathbb{S}^2) : \|h\|_{H^{k+2}} \ll 1, \mathcal{V}(h) = \mathcal{C}_3(h) = 0\}.$$

Its tangent space at the sphere is

$$T_0\mathcal{M}^{k+2} = \left\{ \xi \in H^{k+2}(\mathbb{S}^2) : \int_{\mathbb{S}^2} \xi \, dS = 0, \int_{\mathbb{S}^2} \xi Z \, dS = 0 \right\} =: H_{\text{lin}}^{k+2}.$$

The implicit function theorem gives the local chart

$$\Theta : \mathcal{U} \subset T_0\mathcal{M}^{k+2} \longrightarrow \mathcal{M}^{k+2}, \quad h = \Theta(\xi) = \xi + c(\xi) + d(\xi)Z.$$

2. Spherical graphs

For each h , solve the exterior Neumann problem for ϕ_h and set

$$\mathcal{B}(h, \alpha, V) = \Pi_h(H_h \circ \Psi_h - F_{\alpha, V}(y, \nabla \phi_h) \circ \Psi_h)$$

Projection to remove the unknown Bernoulli constant c .

$$\Pi_h f = f - \frac{\int_{\mathbb{S}^2} f(1+h)^2 dS}{\int_{\mathbb{S}^2} (1+h)^2 dS}$$

Bernoulli map

$$\mathcal{B} : H_{\text{lin}}^{k+2} \times \mathbb{R}^2 \longrightarrow H_{\text{lin}}^k$$

Goal: find (h, α, V) such that $\mathcal{B}(h, \alpha, V) = 0$

3. Linearisation at the sphere

Spherical harmonics Y_ℓ^m :

$$-\Delta_{\mathbb{S}^2} Y_\ell^m = \ell(\ell + 1) Y_\ell^m, \quad \ell \in \mathbb{N}_0, \quad |m| \leq \ell.$$

Lemma

$$D_h \mathcal{B}(0, \alpha, 0) Y_\ell^m = \begin{cases} \lambda_{\ell m}(\alpha) Y_\ell^m, & \ell \geq 1, \\ 0, & \ell = 0. \end{cases}$$

$$\lambda_{\ell m}(\alpha) = \underbrace{\ell(\ell + 1) - 2}_{\substack{\text{mean curvature} \\ -(\Delta_{\mathbb{S}^2} + 2)}} - \underbrace{\alpha^2 \frac{m^2}{\ell + 1}}_{\text{fluid rotation}}$$

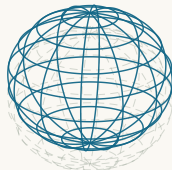
4. Kernel I: constraints remove two modes

$Y_0^0 = 1$, $Y_1^0 = Z$ are kernel modes for every α .

1



Z



fixed volume: $\int_{\mathbb{S}^2} \xi \, dS = 0$

removes the radial direction

fixed vertical centroid: $\int_{\mathbb{S}^2} \xi Z \, dS = 0$

removes vertical translation

5. Axisymmetric bubbles

Restrict to $h = h(Z)$. Then only spherical harmonics with $m = 0$ are admissible. Linearisation is invertible:

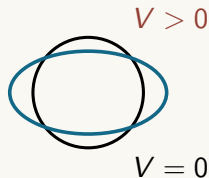
$$D_h \mathcal{B}(0, \alpha, 0) Y_\ell^0 = (\ell(\ell + 1) - 2) Y_\ell^0, \quad \ell \geq 2$$

Theorem (Meyer–N.–Seis, 2025)

For $0 < V \ll 1$, there is a locally unique smooth translating bubble of volume $4\pi/3$, axisymmetric and symmetric about $Z = 0$.

$$h_{0,V}^{\text{ax}} = -\frac{3V^2}{16} \frac{3Z^2 - 1}{2} + O_{H^{k+2}}(V^4).$$

In particular, $\mathcal{B}(h_{0,V}^{\text{ax}}, \alpha, V) = 0$ for every α .



The oblate deformation was predicted by Moore (1965).

6. Kernel II

Consider now $\alpha = \alpha_* = 2\sqrt{3}$.

$$\lambda_{21}(\alpha_*) = 0$$

degree 2, order 1

$$K_1 = \text{span}\{XZ, YZ\}$$



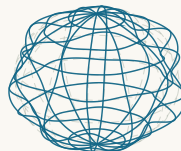
desired bifurcation direction

grey: sphere

$$\lambda_{76}(\alpha_*) = 0$$

degree 7, order 6

$$K_6 = \text{span}\left\{ \begin{array}{l} Z \operatorname{Re}(X + iY)^6, \\ Z \operatorname{Im}(X + iY)^6 \end{array} \right\}$$



unwanted resonances

blue: kernel direction

7. Bernoulli as the gradient of the action

$$\mathcal{E}(\Omega, \alpha, V) = |\partial\Omega| - \frac{1}{2} \int_{\Omega^{\text{out}}} \left| \nabla \phi_{\Omega}^{\alpha, V} \right|^2 dy$$

On the constrained chart $h = \Theta(\xi)$, set

$$\mathcal{J}(\xi, \alpha, V) := \mathcal{E}(\Omega_{\Theta(\xi)}, \alpha, V), \quad \mathcal{F}(\xi, \alpha, V) := \nabla_{\xi}^{\text{L}^2} \mathcal{J}(\xi, \alpha, V).$$

Lemma

We have

$$\mathcal{B}(\Theta(\xi), \alpha, V) = 0 \quad \Longleftrightarrow \quad \mathcal{F}(\xi, \alpha, V) = 0.$$

Moreover, their kernels and invertibility properties are equivalent.

8. Lyapunov–Schmidt reduction

P_j : L^2 -projection onto K_j , $j = 1, 6$.

$$Q := \text{Id} - P_1 - P_6.$$

$$x = a + u, \quad a \in K_1, \quad u \in K_6.$$

Range equation

$$Q\mathcal{F}(x + w(x, \alpha, V), \alpha, V) = 0, \quad w(x, \alpha, V) \in (K_1 \oplus K_6)^\perp.$$

Uniquely solvable for $w = w(x, \alpha, V)$ by the implicit function theorem.

Kernel equations

$$\Phi_j(a, u, \alpha, V) := P_j\mathcal{F}(x + w(x, \alpha, V), \alpha, V), \quad j = 1, 6.$$

$$\Phi_1 = 0, \quad \Phi_6 = 0$$

Four scalar equations remain.

9. Symmetry and equivariance

Recall: R_θ is the rotation by θ about e_3 . Define

$$(\mathcal{U}_\theta h)(\omega) := h(R_{-\theta}\omega), \quad \omega = (X, Y, Z) \in \mathbb{S}^2.$$

We study this action on $K_1 = \text{span}\{XZ, YZ\}$:

$$\mathcal{U}_\theta(XZ) = \cos \theta XZ + \sin \theta YZ,$$

$$\mathcal{U}_\theta(YZ) = -\sin \theta XZ + \cos \theta YZ.$$

On K_j , $j = 1, 6$,

$$\mathcal{U}_\theta|_{K_j} = \cos(j\theta) \text{Id} + \sin(j\theta) J, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

10. $4 \mapsto 3$

Consider the reduced action

$$\hat{\mathcal{J}}(a, u) := \mathcal{J}(a + u + w(a + u, \alpha, V), \alpha, V)$$

$$(\Phi_1, \Phi_6) = \nabla_{(a, u)} \hat{\mathcal{J}}, \quad \hat{\mathcal{J}}(\mathcal{U}_\theta a, \mathcal{U}_\theta u) = \hat{\mathcal{J}}(a, u).$$

$$0 = \left. \frac{d}{d\theta} \hat{\mathcal{J}}(\mathcal{U}_\theta a, \mathcal{U}_\theta u) \right|_{\theta=0} = \langle \Phi_1, Ja \rangle + 6 \langle \Phi_6, Ju \rangle.$$

For $a \neq 0$, the vectors a and Ja span K_1 , hence

$$\Phi_1 = \Phi_6 = 0 \iff \langle \Phi_1, a \rangle = 0, \quad \Phi_6 = 0.$$

11. Moving off the sphere lifts the resonance

At $(a, u) = (0, 0)$,

$$D_{(a,u)} \begin{pmatrix} \Phi_1 \\ \Phi_6 \end{pmatrix} (0, 0, \alpha, V) = \begin{pmatrix} \mu_1(\alpha, V)I_{K_1} & 0 \\ 0 & \mu_6(\alpha, V)I_{K_6} \end{pmatrix}$$

11. Moving off the sphere lifts the resonance

We need to expand

$$D_h \mathcal{B}(h_{0,V}^{\text{ax}}, \alpha, V) = L_0^\alpha + V L_1^{\alpha*} + V^2 L_2^{\alpha*} + O(V^3)$$

uniformly for $|\alpha - \alpha_*| \lesssim V^2$.

$L_0^\alpha Y_\ell^m = \lambda_{\ell m}(\alpha) Y_\ell^m$ diagonal in spherical harmonics

$L_1^{\alpha*}$ preserves m and couples ℓ only to $\ell \pm 1$

$L_2^{\alpha*}$ only its diagonal matrix elements are needed

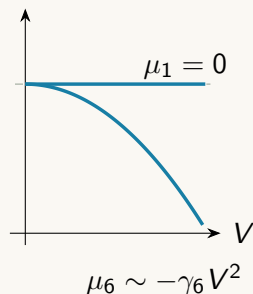
11. Moving off the sphere lifts the resonance

Uniformly for $|\alpha - \alpha_*| \lesssim V^2$,

$$\mu_1(\alpha, V) = \lambda_{21}(\alpha) - \frac{867}{980} V^2 + O(V^3),$$

$$\mu_6(\alpha, V) = \lambda_{76}(\alpha) - \frac{4174947}{284240} V^2 + O(V^3),$$

$$\partial_\alpha \mu_1(\alpha_*, 0) = \partial_\alpha \lambda_{21}(\alpha_*) = -\frac{2\alpha_*}{3} \neq 0.$$



Implicit function theorem to choose $\alpha_1(V)$ with

$$\mu_1(\alpha_1(V), V) = 0, \quad \alpha_1(V) = 2\sqrt{3} - \frac{2601}{1960} \frac{V^2}{2\sqrt{3}} + O(V^3).$$

The resonance is lifted, but inversion costs V^{-2} :

$$\mu_6(\alpha_1(V), V) = -\gamma_6 V^2 + O(V^3), \quad \gamma_6 = \frac{38228049}{13927760} > 0.$$

12. Nonlinearities

Consider now $a = sXZ$ and $\alpha = \alpha_1(V)$.

$$\Phi_1(sXZ, u) = \mu_1(\alpha, V)sXZ + O(|s|^3 + |s|^5 \|u\| + |s| \|u\|^2),$$

$$\Phi_6(sXZ, u) = \mu_6(\alpha, V)u + O(s^6 + s^2 \|u\| + \|u\|^2).$$

Compensates for the V^{-2} loss:

$$\left\| \mu_6(\alpha_1(V), V)^{-1} \Phi_6(sXZ, 0) \right\| \lesssim \frac{|s|^6}{V^2}.$$

12. Nonlinearities

Write the coefficient pairs in K_1 and K_6 as complex numbers z and q .
A rotation through θ acts as

$$z \longmapsto e^{i\theta} z, \quad q \longmapsto e^{6i\theta} q.$$

$$\Phi_6(e^{i\theta} z, e^{6i\theta} q) = e^{6i\theta} \Phi_6(z, q)$$

Set $q = 0$. A Taylor monomial $z^p \bar{z}^r$ can occur only if

$$p - r = 6 \implies p + r \geq 6.$$

Hence,

$$\Phi_6(z, 0) = O(|z|^6)$$

13. Blow-up

We rescale as

$$s = Vt, \quad u = V^2U, \quad \alpha = \alpha_1(V) + V^2\beta.$$

$$\tilde{\Phi}_1 = \frac{\langle \Phi_1, XZ \rangle}{V^3 t \|XZ\|_{L^2}^2}, \quad \tilde{\Phi}_6 = \frac{\Phi_6}{V^4}$$

Both Φ_j are evaluated at $(VtXZ, V^2U, \alpha_1(V) + V^2\beta, V)$.

Equivariance makes the rescaled equations smooth through $Vt = 0$.

13. Blow-up

At $(t, U, \beta, V) = (0, 0, 0, 0)$,

$$D_{(\beta, U)}(\tilde{\Phi}_1, \tilde{\Phi}_6) = \begin{pmatrix} -2\alpha_*/3 & 0 \\ 0 & -\gamma I_{K_6} \end{pmatrix}.$$

Implicit function theorem gives $\beta_*(t, V)$ and $U_*(t, V)$ so that

$$\tilde{\Phi}_j(t, U_*(t, V), \beta_*(t, V), V) = 0, \quad j = 1, 6.$$

Undoing the rescaling gives, for $0 < V < V_0$ and $|s| \leq \eta V$, $u(s, V)$ and $\alpha(s, V)$ with

$$\|u(s, V)\| \lesssim \frac{|s|^6}{V^2}, \quad \alpha(s, V) = \alpha_1(V) + O(s^2).$$

$$\xi(s, V) = sXZ + u(s, V) + w(sXZ + u(s, V), \alpha(s, V), V)$$

$$\mathcal{F}(\xi(s, V), \alpha(s, V), V) = 0$$

14. The centroid follows a true helix

Calculating the asymptotic expansion yields

$$\alpha(s, V) = 2\sqrt{3} - \frac{2601}{1960} \frac{V^2}{2\sqrt{3}} + O(s^2 + V^3).$$

$$\begin{aligned} h(s, V) = & -\frac{3V^2}{16} \frac{3Z^2 - 1}{2} + sXZ - \frac{9}{20\sqrt{3}} VsY \\ & - \frac{5\sqrt{3}}{7} VsY \left(Z^2 - \frac{1}{5} \right) + O_{H^{k+2}} \left(s^2 + |s| V^2 + V^4 \right). \end{aligned}$$

14. The centroid follows a true spiral

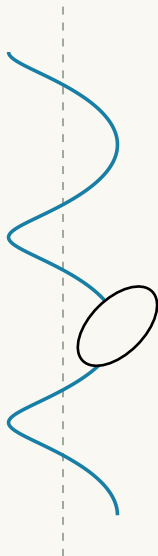
$$\bar{y}(s, V) = \frac{1}{|\Omega(s, V)|} \int_{\Omega(s, V)} y \, dy.$$

$$\pi_{e_3^\perp} \bar{y}(s, V) = -\frac{9}{20\sqrt{3}} Vs e_2 + O(s^2 + |s| V^2).$$

$$0 < |s| \leq \eta V \quad \implies \quad \left| \pi_{e_3^\perp} \bar{y}(s, V) \right| > 0.$$

$$\bar{x}(t) = \mathcal{R}_{\alpha t} \bar{y}(s, V) + Vte_3.$$

Hence, the centroid follows a genuine helix.



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