Kinetic maximal L^p-regularity

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Kolmogorov equation

Interested in solutions $u = u(t, x, v) \colon [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\beta/2} u + f \\ u(0) = g. \end{cases} \tag{1}$$

with data f, g and $\beta \in (0, 2]$.

Key points:

- Studied first by Kolmogorov in 1934 ($\beta = 2$).
- The transport operator $\partial_t + v \cdot \nabla_x$ is called kinetic term.
- Degenerate - Laplacian acts in half of the variables.
- Unbounded coefficient in front of the lower order term.
- Prototype for the Boltzmann equation.

Motivation - Particle Physics

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Particles at position x with velocity v. We describe the movement of the particles with the SDE

$$\begin{cases} \mathrm{d}X(t) = V(t)\mathrm{d}t \\ \mathrm{d}V(t) = \mathrm{d}W(t), \end{cases}$$

where $(W(t))_{t\geq 0}$ is the Wiener process. $\rightsquigarrow$ Kolmogorov equation $\beta = 2$.

The Boltzmann equation models the particle collision, i.e. the change of velocity, more precisely.

Boltzmann equation

The Boltzmann equation can be written as

$$\partial_t u + v \cdot \nabla_x u = Q(u, u) + \mathrm{l.o.t.},$$

where

$$Q(u, g) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t, x, v + h) - u(t, x, v)}{|h|^{n+\beta}} m(g)(t, x, v, h) \mathrm{d}h$$

with

$$m(g)(t, x, v, h) = \int_{w \perp h} g(t, x, v + w) \, |w|^{\gamma+\beta+1} \, \mathrm{d}w$$

and $\beta \in (0, 2)$, $\gamma > -n$ depend on physical assumptions. For fixed g the operator $Q(u, g)$ is the fractional Laplacian in velocity with variable density.

Maximal regularity

General Principle: Find a function space Z for the solution u, a function space X for the inhomogeneity f and a function space X_γ for the initial value g such that the equation admits a unique solution $u \in Z$ if and only if $f \in X$ and $g \in X_\gamma$.

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Example - Heat equation

For all $p \in (1, \infty)$ the heat equation

$$\begin{cases} \partial_t u = \Delta u + f \\ u(0) = g \end{cases}$$

admits a unique solution
$u \in Z = H^{1,p}((0,\infty); L^p(\mathbb{R}^n)) \cap L^p((0,\infty); H^{2,p}(\mathbb{R}^n))$ if and only if

- $f \in X = L^p((0,\infty); L^p(\mathbb{R}^n))$,
- $g \in X_\gamma = B^{2-2/p}_{pp}(\mathbb{R}^n)$ (Besov space).

Moreover, $u \in C([0,\infty); B^{2-2/p}_{pp}(\mathbb{R}^n))$.

Towards kinetic maximal regularity

Which is the right choice for the solution space Z?

For simplicity $\beta = 2$*, every result holds true for* $\beta \in (0, 2)$*.*

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Singular integral theory on homogeneous groups developed by Folland and Stein 1974 allows to prove the following. If $f \in L^p(\mathbb{R}; L^p(\mathbb{R}^{2n}))$, then the solution u of the Kolmogorov equation satisfies

$$\|\partial_t u + v \cdot \nabla_x u\|_p + \|\Delta_v u\|_p \lesssim \|f\|_p \, .$$

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No control of the time-derivative. We prove classical maximal L^p-regularity is not applicable.
Our choice of function space Z:

$$Z = \{u \colon u, \Delta_v u, \partial_t u + v \cdot \nabla_x u \in L^p((0, T); L^p(\mathbb{R}^{2n}))\}.$$

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Divide and conquer

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Classical Method: L^p-estimates, singular integrals,...

Done, Folland/Stein.

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Homogeneous eq. with non-zero intial-value

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u \\ u(0) = g \end{cases}$$

Classical Method: Studying the trace space of Z.
TODO!

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The trace space of Z - 1

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Sketch of the proof

Define

$$[\Gamma u](t,x,v) = u(t, x+tv, v) \text{ and } [\Gamma(t)w](x,v) = w(x+tv, v)$$

on functions $u\colon [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ and $w\colon \mathbb{R}^{2n} \to \mathbb{R}$.

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on functions $u\colon [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ and $w\colon \mathbb{R}^{2n} \to \mathbb{R}$. Then,

$$\partial_t \Gamma u = \Gamma(\partial_t u + v \cdot \nabla_x u).$$

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If $u \in Z$, then $\Gamma u \in H^{1,p}((0,T); L^p(\mathbb{R}^{2n}))$, whence

$$\Gamma u \in C([0,T]; L^p(\mathbb{R}^{2n})).$$

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As $(\Gamma(t))_{t\in\mathbb{R}}$ is C_0-group, it follows

$$u = \Gamma^{-1}(t)\Gamma(t)u \in C([0,T]; L^p(\mathbb{R}^{2n})).$$

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$$u = \Gamma^{-1}(t)\Gamma(t)u \in C([0,T]; L^p(\mathbb{R}^{2n})).$$

Consequently, $\mathrm{Tr}(Z)$ well-defined and $Z \hookrightarrow C([0,T]; \mathrm{Tr}(Z))$.

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The trace space of Z cannot be characterized by classical interpolation theory. As for the heat equation we expect

$$\mathrm{Tr}(Z) \hookrightarrow B^{2-2/p}_{pp,v}(\mathbb{R}^{2n}).$$

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Is there any control of regularity in x? Yes!

Towards kinetic maximal regularity

Regularity transfer from v to x.

The phenomenon of regularity transfer.

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Regularity transfer from v to x.

The phenomenon of regularity transfer.

Theorem (Bouchut 2002)

Let $u \in L^p((0,T); L^p(\mathbb{R}^{2n}))$ with
$\partial_t u + v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))$ and $u \in L^p((0,T); H_v^{2,p}(\mathbb{R}^{2n}))$,
then
$$u \in L^p((0,T); H_x^{2/3,p}(\mathbb{R}^{2n})).$$

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Theorem (Bouchut 2002)

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$$u \in L^p((0,T); H_x^{2/3,p}(\mathbb{R}^{2n})).$$

In words: If u is the solution of a kinetic equation and u has two derivatives in velocity we obtain $2/3$ of a derivative in space, too. Very useful and powerful result!

Towards kinetic maximal regularity

The initial value problem - 2

Similar to Bouchut we also get some regularity in x for the trace space.

Theorem (N., Zacher, 2020)

Let $p \in (1, \infty)$, then

$$\mathrm{Tr}(Z) \cong B^{2/3(1-1/p)}_{pp,x}(\mathbb{R}^{2n}) \cap B^{2-2/p}_{pp,v}(\mathbb{R}^{2n})$$

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Proof

Littlewood-Paley decomposition, Fourier analysis and the fundamental solution for the Kolmogorov equation.

Kinetic maximal L^p-regularity

for the (fractional) Kolmogorov equation

Theorem (N., Zacher, 2020)

Let $T \in (0,\infty)$. For all $p \in (1,\infty)$ the Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

admits a unique solution $u \in Z$ if and only if

- $f \in X = L^p((0,T); L^p(\mathbb{R}^n))$,
- $g \in X_\gamma = B^{2/3(1-1/p)}_{pp,x}(\mathbb{R}^{2n}) \cap B^{2-2/p}_{pp,v}(\mathbb{R}^{2n})$.

Moreover, $u \in C([0,T]; X_\gamma)$.

We say the operator $A = \Delta_v$ admits kinetic maximal L^p-regularity.

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Change of base space

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- We also consider the case $X = L^q(\mathbb{R}^{2n})$ for some $q \in (1, \infty)$ different from p and prove kinetic maximal $L^p(L^q)$-regularity.
- For $p \in (1, \infty)$, $q = 2$ we characterize weak solutions to the fractional Kolmogorov equation.

Extensions

Temporal weights

Instead of $L^p((0,T);X)$ we consider a Lebesgue space with temporal weight of the form $t^{1-\mu}$ for some $\mu \in (1/p, 1]$ defined as

$$L^p_\mu((0,T);X) = \{u\colon (0,T) \to X\colon \int_0^T t^{p-p\mu} \, \|u(t)\|_X^p \, \mathrm{d}t < \infty\}.$$

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Key features:

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- The trace space of Z_μ is given by $\mathrm{Tr}(Z_\mu) = X_{\gamma,\mu} = B^{2/3(\mu-1/p)}_{pp,x}(\mathbb{R}^{2n}) \cap B^{2(\mu-1/p)}_{pp,v}(\mathbb{R}^{2n})$.
- Instantaneous regularization $Z_\mu(0,T) \hookrightarrow Z(\delta,T) \hookrightarrow C([\delta,T];X_{\gamma,1})$ for all $\delta > 0$.

Extensions

Kinetic maximal $L^p_\mu(L^q)$-regularity

Theorem (N., Zacher, 2020)

Let $T \in (0, \infty)$. For all $p, q \in (1, \infty)$ and any $\mu \in (1/p, 1]$ the Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

admits a unique solution

$$u \in Z_\mu = \{u\colon u,\, \Delta_v u,\, \partial_t u + v \cdot \nabla_x u \in L^p_\mu((0, T); L^q(\mathbb{R}^{2n}))\}.$$

if and only if

- $f \in X = L^p_\mu((0, T); L^q(\mathbb{R}^n))$,
- $g \in X_{\gamma,\mu} = B^{2/3(\mu-1/p)}_{qp,x}(\mathbb{R}^{2n}) \cap B^{2(\mu-1/p)}_{qp,v}(\mathbb{R}^{2n})$.

Moreover, $u \in C([0, T]; X_{\gamma,\mu})$.

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- $Au = -(-\Delta_v)^{\frac{\beta}{2}} u$ with $\beta \in (0,2)$
- non-local integro-differential operators acting in velocity with possibly time, space and velocity dependent density

Extensions

Different Operators

Theorem (N., Zacher, 2020)

Let $p, q \in (1, \infty)$, $\mu \in (1/p, 1]$, $a \in L^\infty([0,T] \times \mathbb{R}^{2n}; \mathrm{Sym}(n))$, $b \in L^\infty([0,T] \times \mathbb{R}^{2n}; \mathbb{R}^n)$ and $c \in L^\infty([0,T] \times \mathbb{R}^{2n}; \mathbb{R})$. If $a \geq \lambda \mathrm{Id}$ for some $\lambda > 0$ and if the function $(t, x, v) \mapsto a(t, x + tv, v)$ is uniformly continuous, then then the family of operators

$$A(t)u = a(t, \cdot) \colon \nabla_v^2 u + b(t, \cdot) \cdot \nabla_v u + c(t, \cdot) u$$

admits kinetic maximal $L^p_\mu(L^q)$-regularity.

Quasilinear kinetic diffusion problem

Short-time existence

We prove short-time existence of strong L^p_μ-solutions to the following quasilinear kinetic diffusion equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \nabla_v \cdot (a(u) \nabla_v u) \\ u(0) = g \end{cases}$$

for $a \in C^2_b(\mathbb{R}; \mathrm{Sym}(n))$ with $a \geq \lambda Id$ for some $\lambda > 0$, $\mu - 1/p > 2n/p$ and $g \in X_{\gamma,\mu}$.

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Methods: Freeze the equation at the initial value and use kinetic maximal L^p-regularity for the frozen equation. Here, we need the maximal regularity of $A = a(g(x,v)) \colon \nabla_v^2 u$.

Further research

Possible directions:

- weak L^p-solutions
- study kinetic quasilinear problems from physics/economics/biology
- regularity and long-time behavior of solutions to quasilinear problems
- a priori-estimates to solutions of the Kolmogorov equation
- conditions on the operator A such that it admits kinetic maximal L^p-regularity
- boundary problems
- different first order terms, for example $\partial_t + \langle x, B\nabla \rangle$ or the relativistic kinetic term $\partial_t + \frac{v}{\sqrt{1+|v|^2}}\nabla_x$

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Discussion

A quasilinear heat equation

We want to understand if the solution constructed for the kinetic quasilinear diffusion problem exist for all positive times. Not even known in the parabolic case. We investigate

$$\begin{cases} \partial_t u = \nabla \cdot (a(u)\nabla u) = a(u)\Delta u + a(u) \left|\nabla u\right|^2 \\ u(0) = g \end{cases} \tag{2}$$

for $a \in C^2(\mathbb{R}, (\lambda, K))$ and $0 < \lambda < K$.

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Theorem

If $p > n + 2$, then for all $g \in B_{pp}^{2-2/p}(\mathbb{R}^n)$ there exists a time $T = T(g)$ and function $u \in Z := H^{1,p}((0, T); L^p(\mathbb{R}^n)) \cap L^p((0, T); H^{2,p}(\mathbb{R}^n))$, which solves the quasilinear diffusion equation (2).

Discussion

Long time existence for small initial datum - 1

We assume that $\|g\|_\infty \leq \varepsilon < 1$. Recall $B_{pp}^{2-2/p} \subset L^\infty(\mathbb{R}^n)$ for $p > n + 2$.

Step 0: Extend the local solution to a maximal interval of existence $[0, t^+)$. Let $T < t^+$.

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Step 1: Maximum principle. Let $g \in X_\gamma = B_{pp}^{2-2/p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$. If u is an L^p-solution to $\partial_t u = \nabla \cdot (a(t,x)\nabla u)$, $u(0) = g$, on $[0, T]$ then

$$\inf_{x\in\mathbb{R}^n} g(x) \leq u(t,x) \leq \sup_{x\in\mathbb{R}^n} g(x).$$

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Step 2: A priori estimate. There exists constants $C = C(n, \lambda, K)$ and $\alpha = \alpha(n, \lambda, K) > 0$ such that

$$\|u\|_{C^\alpha([0,T]\times\mathbb{R}^n)} \leq C.$$

Discussion
Long time existence - 2

Step 3: We freeze the nonlinearity, i.e. consider the linear problem

$$\begin{cases} \partial_t w = b(t,x)\Delta w \\ u(0) = g \end{cases}$$

with $b(t,x) = a(u(t,x))$. Due to the uniform continuity proven in Step 2 there exists a constant $M = M(\lambda, n, K) > 0$, independent of T, such that

$$\|w\|_{Z(0,T)} \leq M \|g\|_{X_\gamma} .$$

Discussion

Long time existence - 3

Step 4: We have

$$\begin{aligned} \|u\|_{Z(0,T)} &\le \|u-w\|_{Z(0,T)} + \|w\|_{Z(0,T)} \\ &\le M \left\| |\nabla u|^2 \right\|_p + \|w\|_{Z(0,T)} \\ &\le M \|u\|_\infty \left\|\nabla^2 u\right\|_p + \|w\|_{Z(0,T)} \\ &\le M \|g\|_\infty \|u\|_Z + \|w\|_{Z(0,T)} . \end{aligned}$$

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Long time existence - 3

Step 4: We have

$$\begin{aligned}
\|u\|_{Z(0,T)} &\leq \|u-w\|_{Z(0,T)} + \|w\|_{Z(0,T)} \\
&\leq M \left\| |\nabla u|^2 \right\|_p + \|w\|_{Z(0,T)} \\
&\leq M \|u\|_\infty \left\| \nabla^2 u \right\|_p + \|w\|_{Z(0,T)} \\
&\leq M \|g\|_\infty \|u\|_Z + \|w\|_{Z(0,T)} \,.
\end{aligned}$$

Step 5: If $\varepsilon < \frac{1}{M}$, then

$$\lim_{T \to T^+} \|u\|_{Z(0,T)} < \infty.$$

Consequently, $t^+ = \infty$ (general principle as for ODE).

Discussion

Long time existence - 4

Are there other conditions on the initial value, the solution such that $t^+ = \infty$?

- We need to control the L^{2p}-norm of ∇u.
- On bounded sets Hölder continuity implies Sobolev regularity, whence this norm can be controlled by Gagliardo-Nirenberg inequality and we always have global existence.
- L^1-bound on the initial value g? This gives an a priori bound for u in all L^p-norms.
- Other ideas?